Mathematics IV, Exercises 1. Metrics and countable sets.

Corrections April 30.

1) Metrics

a Consider the vector-space ($\mathbb{M}$) of real numbers ($\mathbb{R}$). Show that for $z, z_1 \in \mathbb{R}$ the function $d : \mathbb{M} \times \mathbb{M} \to \mathbb{R}^+$ defined as

$$d(z, z_1) = f(|z - z_1|), \quad f : \mathbb{R}^+ \to \mathbb{R}^+$$

is a metric provided that for $x, y \in \mathbb{R}^+$

A) $f(0) = 0$

B) $f(x) > 0$ for $x > 0$

C) $f(y) \geq f(x)$ for $y \geq x$. i.e. f is a monotonically growing function.

D) $f(x)/x \geq f(y)/y$ for $y \geq x$.

b Show that given conditions D) and A) $f(x)$ is a concave function. Note: if $f(x)$ is twice differentiable concave means $f''(x) < 0$!

c Now show using **a** that:

$$f(x) = \frac{x}{1+x} \quad \text{and} \quad f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}$$

induce metrics.

2 Countable (in German *abzählbar*) sets.

Def A set $\mathbb{S}$ is called finite if there exists a one-to-one mapping (i.e. bijective mapping or a bijection) between $\mathbb{S}$ and the the set $\mathbb{N}_n = \{0, 1, 2, 3, \cdots, n-1\}$ for some natural number n. A set is called countably infinite if there exists a bijective mapping between $\mathbb{S}$ and the set $\mathbb{N} = \{0, 1, 2, \cdots\}$ of all natural numbers.

a Show that the set $\mathbb{S} = \{a, b, c\}$ is countable.

b Show that the set of integer numbers $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$ is countable infinite.

c Show that $\mathbb{S} = \mathbb{N} \times \mathbb{N} = \{x | x = (p, q), p, q \in \mathbb{N}\}$ is countable infinite.

d Can you now argue that the set of rational numbers, $\mathbb{Q} = \{x | x = p/q, p \in \mathbb{Z}, q \in \mathbb{N}, q \neq 0\}$ is countable infinite.