Mathematics IV, Exercises 2. Tensor product of Hilbert spaces. Operators.

Corrections May 7.

1) Tensor product of Hilbert spaces.

Consider the Hilbert $\mathcal{H}^{(1/2)}$ space on the $\mathbb{C}$ spanned by the vectors $\mathbf{u} = (\mathbf{1}, \mathbf{0})$ and $\mathbf{d} = (\mathbf{0}, \mathbf{1})$ and with scalar product satisfying $\langle u|u\rangle = \langle d|d\rangle = 1$ and $\langle d|u\rangle = 0$.

a Write an orthonormal basis for the tensor space: $\mathcal{H} = \mathcal{H}^{(1/2)} \otimes \mathcal{H}^{(1/2)}$.

b Consider the linear operator $P : \mathcal{H} \to \mathcal{H}$

$$Pf = P\left(f_1 \otimes f_2\right) = f_2 \otimes f_1.$$

Here $f \in \mathcal{H}$ and $f_1, f_2 \in \mathcal{H}^{(1/2)}$. Show that, $P^2 = 1$, and $P^\dagger = P$ (that means $\langle f|Pg\rangle = \langle Pf|g\rangle \;\; \forall f, g \in \mathcal{H}$).

c Consider the Hilbert spaces:

$$\mathcal{H}_S = \{f | f \in \mathcal{H} \;\text{ and }\; Pf = f\}$$
$$\mathcal{H}_A = \{f | f \in \mathcal{H} \;\text{ and }\; Pf = -f\}$$

Write an orthonormal basis for $\mathcal{H}_S$ and $\mathcal{H}_A$.

d Show that $\mathcal{H}_S \perp \mathcal{H}_A$.

e Consider $\tilde{\mathcal{H}} = \mathcal{H}_A \oplus \mathcal{H}_S$. Write a basis of $\tilde{\mathcal{H}}$ and show that there is a unitary mapping between the elements of $\tilde{\mathcal{H}}$ and those of $\mathcal{H}$. That is $\mathcal{H} = \mathcal{H}^{(1/2)} \otimes \mathcal{H}^{(1/2)}$ is isomorph to $\mathcal{H}_A \oplus \mathcal{H}_S$.

2 Consider the operators A and B with $[A, B] \equiv AB - BA \neq 0$ but $[[A, B], A] = [[A, B], B] = 0$. Show that:

$$e^A e^B = e^{A+B} e^{[A,B]/2}$$