Mathematics IV, Exercises 3.

Corrections May 14.

1) Consider a Hilbert space $\mathcal{H}$. Let $f_1, f_2 \in \mathcal{H}$. Furthermore, assume that f_1 and f_2 are linearly independent.

a. Build an orthonormal basis of the sub-space of $\mathcal{H}$ spanned by f_1 and f_2.

b. Can you now generalize this result and build an orthogonal basis of the space spanned by the linearly independent vectors $f_1, f_2 \cdots f_n$.

c. Consider the space of real valued polynomials in the interval $[0, 1]$ with scalar product $\langle f|g\rangle = \int_0^1 \mathrm{d}x f(x)g(x)$. Let $g(x) = x^2$ and $f(x) = x^3$ and show that the vectors g and f are linearly independent. Build an orthonormal basis of the space spanned by g and f.

2) The two dimensional Hilbert space of spin-1/2 degree of freedom is spanned by the two vectors:

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and with scalar product satisfying $\langle u|u\rangle = \langle d|d\rangle = 1$ and $\langle d|u\rangle = 0$.

a. The Pauli spin matrices are operators on this two-dimensional space and are defined by:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

such that $\sigma_z \mathbf{u} = \mathbf{u}$ and $\sigma_z \mathbf{d} = -\mathbf{d}$. Show that the operators are self-adjoint and that they satisfy the commutation relation: $[\sigma_x, \sigma_y] = 2i\sigma_z$.

b Find the eigenvalues and eigenvectors of σ_x.

c A spin degree of freedom in a magnetic field B oriented along the x-direction satisfies the Schrödinger equation:

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{\Psi}(t) = \mu_B B \sigma_x \mathbf{\Psi}(t)$$

where μ_B is the Bohr magneton and $\mathbf{\Psi(t)}$ the wave function (i.e. vector in the Hilbert space). Solve the above differential equation given the initial condition $\mathbf{\Psi}(t = 0) = \mathbf{u}$.

d Compute the scalar product: $\langle \mathbf{u}|\mathbf{\Psi}(t)\rangle$.

e Compute $(\sigma_x)^n$ for $n \in \mathbb{N}$. Use this result to calculate:

$$U(t) = e^{-it\mu_B B\sigma_x/\hbar}.$$

Show that $U(t)$ is a unitary operator and that $\mathbf{\Psi}(t) = U(t)\mathbf{u}$.

3) Consider the operators A, B, C on the Hilbert space $\mathcal{H}$.

a Show that

$$
\begin{aligned}
&(1)\ \ ||A+B|| \leq ||A|| + ||B|| \\
&(2)\ \ ||AB|| \leq ||A|| ||B|| \\
&(3)\ \ [A, BC] = [A, B]C + B[A, C] \\
&(4)\ \ ||[A, B]|| \leq 2||A||\ ||B|| \\
&(5)\ \ ||[A, B^n]|| \leq 2||A||\ ||B^{n-1}|| ||B||
\end{aligned}
$$

b Let A and B be bounded. Show that the equation $[A, B] = i$ can not be correct. Hint: Assume that $[A, B] = i$ and show that $[A, B^n] = inB^{n-1}$. Combine this result with Equation (5) of part **a** to show that there is a contradiction.

Note that this is the type commutation relation which leads to the Heisenberg uncertainty principle.