Mathematics IV, Exercises 4.

Corrections May 21.

1) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces over the field (Körper) of complex numbers. Show that $\mathcal{H}_1 \otimes \mathcal{H}_2$ as well as $\mathcal{H}_1 \oplus \mathcal{H}_2$ are also Hilbert spaces. That is you have to show that they are complete unitary spaces.

2) Distributions: Show that

$$\lim_{\epsilon \to 0} \frac{1}{x + i\epsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x)$$

where $\delta(x)$ is the Dirac δ-function and $\mathcal{P}$ the principle value defined as:

$$\int_{-\infty}^{\infty} \mathrm{d}x \delta(x)\phi(x) = \phi(0) \quad \text{and} \quad \int_{-\infty}^{\infty} \mathrm{d}x \mathcal{P}\frac{1}{x}\phi(x) = \lim_{\epsilon \to 0}\left[\int_{-\infty}^{-\epsilon} \mathrm{d}x \frac{\phi(x)}{x} + \int_{\epsilon}^{\infty} \mathrm{d}x \frac{\phi(x)}{x}\right].$$

Here $\phi(x)$ is in the Schwartz space of test-functions.

3) Let $\mathcal{H}$ be a Hilbert-space and A an operator on this Hilbert space as well as a vector $|f_0\rangle$ (in the Dirac notation) such that:

$$\left[A, A^\dagger\right] = 1, \quad A|f_0\rangle = 0 \quad \text{and} \quad \langle f_0|f_0\rangle = 1.$$

a. Consider

$$H = \omega A^\dagger A. \tag{1}$$

Show that H is hermitian and that $|f_n\rangle = \frac{1}{\sqrt{n!}}\left(A^\dagger\right)^n |f_0\rangle$ are eigenvectors of H. Find the eigenvalues.

b. Show that

$$\langle f_n|f_m\rangle = \delta_{n,m}$$

c. Let $\mathcal{H} = L^2(\mathbb{R})$ with scalar product

$$\langle g|f\rangle = \int_{-\infty}^{\infty} \mathrm{d}x \, [g(x)]^* \, f(x)$$

and

$$A = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + x\right), \quad \text{and} \quad f_0(x) = \frac{e^{-x^2/2}}{\pi^{1/4}}. \tag{2}$$

Show that $\left[A, A^\dagger\right] = 1$ and that $(Af_0)(x) = 0$. Compute the operator H. Note. This is the Hamilton operator of the quantum mechanical harmonic oscillator. The eigenfunctions $f_n(x)$ are the Hermite functions.

d. Do the functions $f_n(x)$ with $f_0(x)$ defined in **c** form a complete orthonormal set of the Hilbert-space. To show completeness you have to show that the only $g(x) \in \mathcal{H}$ which satisfies $\langle g|f_n\rangle = 0 \;\; \forall n$ is $g(x) = 0$. Here are some hints. Show that $\langle g|f_n\rangle = 0 \;\; \forall n$ leads to $\langle g|x^n f_0\rangle = 0$ and in turn to $\langle g|e^{ikx} f_0\rangle = 0$ for all $k \in \mathbb{R}$. Then use the properties of Fourier transforms.

e. Let $L_{f_n}(g) \equiv \langle f_n|g\rangle$ for all $g \in \mathcal{H}$. In the Dirac notation one writes this linear functional as $\langle f_n|$. Show than the completeness of the functions $f_n(x)$ leads to

$$\sum_n |f_n\rangle\langle f_n| = 1$$

and that

$$H = \sum_n E_n |f_n\rangle\langle f_n|$$

where $H|f_n\rangle = E_n|f_n\rangle$.