Mathematics IV, Exercises 6.

Corrections June 4.

1). Distributions.

a Consider the Fermi function:

$$f_\beta(x) = \frac{1}{1 + e^{\beta(x-x_0)}}$$

and calculate

$$\frac{\partial}{\partial x} \lim_{\beta\to\infty} f_\beta(x)$$

b Show that one can write

$$\delta(x - x_0) = \sum_{n=0}^{\infty} a_n \delta^{(n)}(x)$$

where $\delta^{(n)}(x)$ is the n-th derivative of the Dirac δ-function. Determine the values of the coefficients a_n.

c Green function. Consider the differential equation:

$$\Box A(x_0, \vec{x}) = \frac{4\pi}{c} j(x_0, \vec{x}) \text{ with } \Box = \left[\left(\frac{\partial}{\partial x_0}\right)^2 - \sum_{i=1}^{3} \left(\frac{\partial}{\partial x_i}\right)^2 \right] \tag{1}$$

where $\vec{x} = (x_1, x_2, x_3)$. Recall your electrodynamics class: $x_0 = ct$, c is the velocity of light,t the time, A a component of the 4-vector potential and j a component of the 4-current. Show that

$$G(x_0, \vec{x}) = \frac{1}{4\pi r}\delta(x_0 - r) \quad \text{with } \ r = |\vec{x}|$$

satisfies the equation:

$$\left[\left(\frac{\partial}{\partial x_0}\right)^2 - \sum_{i=1}^{3} \left(\frac{\partial}{\partial x_i}\right)^2 \right] G(x_0, \vec{x}) = \delta(x_0)\delta(\vec{x})$$

Here are some hints: You can use the fact that for $\phi(\vec{x}) \in \mathbb{S}(\mathbb{R}^3)$ ($\mathbb{S}$ is the space of test functions) the following identity holds:

$$\int_0^{2\pi} d\alpha \int_0^{\pi} d\gamma \sin(\gamma)\ \Delta\phi(r\vec{e}) = \frac{1}{r}\frac{d^2}{dr^2} r \int_0^{2\pi} d\alpha \int_0^{\pi} d\gamma \sin(\gamma)\ \phi(r\vec{e})$$

Here, $\vec{e} = (\sin(\gamma)\cos(\alpha), \sin(\gamma)\sin(\alpha), \cos(\gamma))$ is a unit vector in $\mathbb{R}^3$ so that the integral runs over the unit sphere. Furthermore, $\Delta = \sum_{i=1}^{3} \left(\frac{\partial}{\partial x_i}\right)^2$. Using this equation, you have to show that: $\int d^4\underline{x} G(\underline{x}) \Box \Phi(\underline{x}) = \Phi(\underline{0})$. $\underline{x} = (x_0, \vec{x})$ and $\Phi \in \mathbb{S}(\mathbb{R}^4)$.

d Using the result of **c** show that:

$$A(\underline{x}) = \frac{4\pi}{c} \int d^4\underline{y} G(\underline{x} - \underline{y}) j(\underline{y})$$

is a solution of equation (1).

2). Branching points (Verzweigungspunkte) branch lines (Verzweigungslinien)

a Assume that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ with convergence radius R. (i.e. the series converges for $|z - z_0| < R$). Show that for all z in the convergence radius R, the function $f(z)$ has no branching points.

b Find the branching points (if any) of the following functions:

1. $f(z) = \ln(z - a)$

2. $f(z) = \cos(z)$

3. $f(z) = 1/z$

4. $f(z) = \sqrt{z^2 + 4}$

c Find a Riemann surface on which $\ln(z - a)$ is uniquely defined.

3). Analytical functions.

Let $f(z)$ be analytical function and $\gamma_1(t)$, $\gamma_2(t)$ two (differentiable) paths in the complex plane which intersect at $t = 0$ ($\gamma_1(t = 0) = \gamma_2(t = 0)$) with angle ϕ. Show that the images of those two paths $\eta_1(t) = f(\gamma_1(t))$ and $\eta_2(t) = f(\gamma_2(t))$ intersect with the same angle at $t = 0$.