Mathematics IV, Exercises 7.

Corrections June 11.

1). Möbius functions.

Consider the complex function

$$f(z) = \frac{az+b}{cz+d} \quad \text{with} \quad ad-bc \neq 0 \quad \text{and} \quad a,b,c,d \in \mathbb{C}$$

which are called Möbius transformations.

a Show that $f(z)$ is analytical in the complete complex plane apart from point where there is a pole.

b Show that the Möbius transformations form a non-abelian group under the composition $(f \cdot g)(z) = f(g(z))$. In doing so, show that a Möbius transformation corresponds to a 2×2 matrix of complex numbers and the composition of two Möbius transformations is nothing but the multiplication of the two corresponding matrices.

c Using the result of **b** show that each Möbius transformation may be decomposed in elementary transformations - dilatations, rotations $D_\alpha(z) = \alpha z$, translations $T_\beta(z) = z + \beta$ and inversion $I_\gamma(z) = \gamma/z$.

d Lines and circles. Show that for $k > 0$, $k \neq 1$, and two complex numbers α, β the set of complex numbers satisfying the equation

$$|z-\alpha| = k|z-\beta|$$

defines a circle. (The circle of Apollonius).

Show that the equation $|z-\alpha| = |z-\beta|$ defines a line in the complex plane.

e Using the above, show that a Möbius transformation transforms a line in the complex plane into a line or a circle in the complex plane.

2 Harmonic functions and analytical continuation.

Let

$$u(x,y) = e^{-x}(x\sin(y) - y\cos(y))$$

a Show that $u(x,y)$ is a Harmonic function. That is $\Delta u = 0$ with $\Delta = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2$

b Find a function $v(x,y)$ such that $f(z) = u(x,y) + iv(x,y)$ is analytical.

c Write $f(z)$ only as a function of z.