Mathematics IV, Exercises 8.

Corrections June 18.

1). Integration

Calculate the following integrals:

a $\int_C (x^2 - iy^2)dz$ along a parabolic curve C from $(1,2)$ to $(2,8)$ specified by $y = 2x^2$

b Same as **a)** but for a line ranging from $(1,2)$ to $(2,8)$. Are the results different? Why?

c $\oint |z|^2 dz$ along the square with corners $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.

d $\oint z^2 dz$ for the same path as in **c)**

2). Kramer-Krönig

Consider a complex function $\chi(z)$ which is analytic in the upper half plane ($\mathrm{Im}(z) > 0$) and which decays for large values of z in the upper half plane as $1/|z|$. Show that

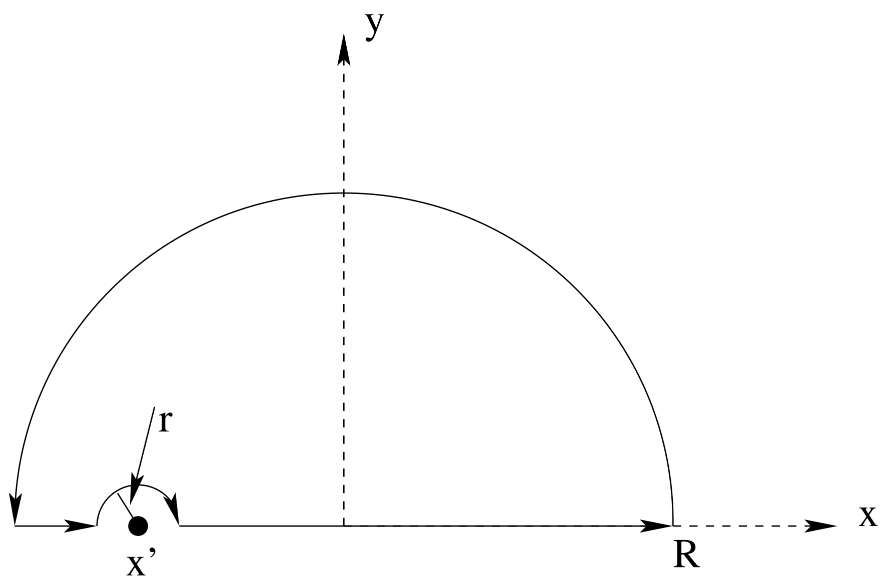
$$\chi(x') = \frac{1}{i\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\chi(x)}{x-x'}\mathrm{d}x \text{ such that}$$

$$\mathrm{Re}\chi(x') = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}\chi(x)}{x-x'}\mathrm{d}x \text{ and } \mathrm{Im}\chi(x') = \frac{-1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\chi(x)}{x-x'}\mathrm{d}x$$

where the principle value is given by:

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{\chi(x)}{x-x'}\mathrm{d}x = \lim_{r\to 0}\left\{\int_{-\infty}^{x'-r}\frac{\chi(x)}{x-x'}\mathrm{d}x + \int_{x'+r}^{\infty}\frac{\chi(x)}{x-x'}\mathrm{d}x\right\}$$

Here is a hint. Consider the path shown in the figure and then take the limits $R \to \infty$ and $r \to 0$.



And here is why this is an important relation in physics. Say that we are interested in the (linear) response of a system to an external perturbation. For instance consider a chunk of copper and turn on a time (t) dependent magnetic field $B(t)$. The induced magnetization ($M(t)$) when the field is very small reads: $M(t) = \int_{-\infty}^{\infty} \chi(t-t')B(t')\mathrm{d}t'$ where χ is called the susceptibility. Causality tells us that there is no reaction of the system *before* the perturbation is turned on. This means that $\chi(t-t') = 0$ for $t - t' < 0$. Hence, the Fourier transform $\chi(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\chi(t)e^{i\omega t} \equiv \int_0^{\infty} \mathrm{d}t\chi(t)e^{i\omega t}$. We can generalize the susceptibility $\chi(\omega)$ to a function of a complex variable $\chi(z) = \int_0^{\infty} \mathrm{d}t\chi(t)e^{izt}$ and show that is satisfies the above conditions.

3 Let $f(z)$ be a continuous function in the simply connected (*einfach zusammenhängend*) domain $\mathcal{G} \in \mathbb{C}$. Furthermore, assume that the integral

$$\int_{z_0}^{z} \mathrm{d}z' f(z')$$

is independent on the integration path from z_0 to z. $(z, z_0 \in \mathcal{G})$. Hence we can define the function: $F(z) = \int_{z_0}^{z} \mathrm{d}z' f(z')$. Show that $F(z)$ is analytical in the domain $\mathcal{G}$ and that $\frac{d}{dz}F(z) = f(z)$.