Mathematics IV, Exercises 9.

Corrections June 25.

1). Use the Cauchy integral theorem to show the following.

a) Consider the function $f(z) = \frac{1}{z-i}$ and show that,

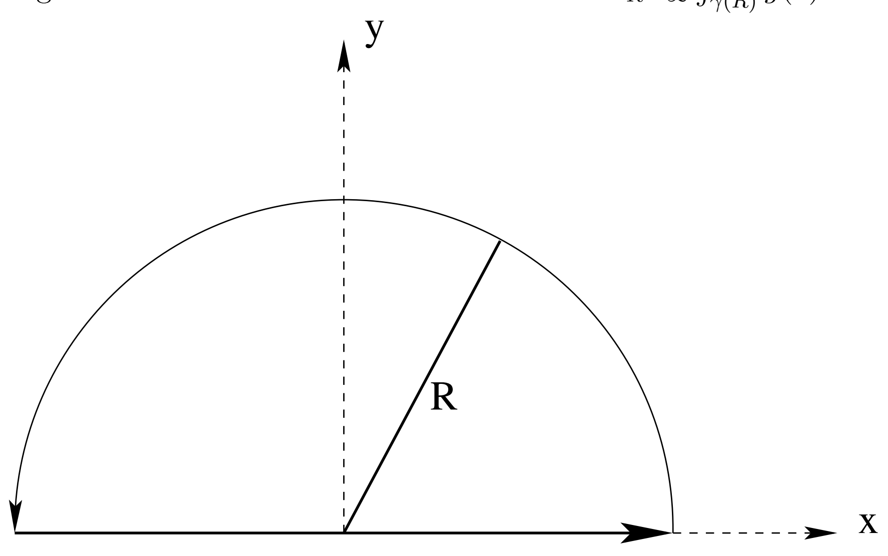
$$\oint_{C_r(z_0=0)} f(z)dz = \begin{cases} \oint_{C_\epsilon(z_0=i)} f(z)dz & \text{if } r > 1 \\ 0 & \text{if } r < 1 \end{cases}$$

In the above, $C_r(z_0)$ denotes a circle centered at z_0 with radius r and $\epsilon > 0$.

b) Now consider $f(z) = \frac{1}{(z-i)(z+i)}$ and show that,

$$\oint_{C_r(z_0=0)} f(z)dz = \begin{cases} \oint_{C_\epsilon(z_0=i)} f(z)dz + \oint_{C_\epsilon(z_0=-i)} f(z)dz & \text{if } r > 1 \\ 0 & \text{if } r < 1 \end{cases}$$

c Consider the function $g(x) = 1/(x^2+1)$ ($g : \mathbb{R} \to \mathbb{R}$). Continue this function to the analytical plane to obtain the function $g(z)$. Let G be the domain in the complex plane where $g(z)$ is analytical. Is $g(z)$ unique in G ? Calculate the integral $\oint_{\gamma(R)} g(z)dz$ for the path shown in the Figure and for all values of R. Show that $\lim_{R\to\infty} \oint_{\gamma(R)} g(z)dz = \int_{-\infty}^{\infty} g(x)dx$.



2. Let $f(z)$ be an analytical function in the domain $|z| \leq 1$. Suppose that we know the values of $f(z)$ on the unit circle (i.e. $|z| = 1$). Given only this information determine the values of $f(z)$ in the unit circle, and write down the Taylor expansion of $f(z)$ around the point $z = 0$. Let us parameterize the unit circle with the angle $\phi \in [0, 2\pi]$ and that on the unit circle the function $f(z)$ takes the values $f(e^{i\phi}) = \cos(m\phi) + i\sin(m\phi)$. Can you determine the function $f(z)$?

3. Show that:

$$\int_0^{2\pi} e^{r\cos\phi}\cos(r\sin\phi - n\phi)d\phi = \frac{2\pi r^n}{n!}$$

Here is a hint. Use the Cauchy integral formula to compute the n^{th} derivative of the exponential function.

4. Let $f(z)$ be an analytical function in the domain G.

(a) Show that from $|f(z)| = M \;\; \forall z \in G$ follows that $f(z) = \alpha \;\; \forall z \in G$ $(\alpha \in \mathbb{C})$. That is: if $|f(z)|$ is a constant then $f(z)$ is also a constant. (Here is a hint. Recall that we have shown that if $f(z)$ is analytical then $\frac{\partial f}{\partial z^*} = 0$)

(b) Assume that $f(z)$ is not equal to a constant. With this assumption show that $|f(z)|$ takes it's maximum value on the boundary of the domain G. (Here is a hint. Assume the contrary and use the Cauchy formula to show that this leads to a contradiction)

With a) and b) you have demonstrated that an analytical function in the domain G which takes it's maximum within the domain G is a constant!