Mathematics IV, Exercises 10.

Corrections July 2.

1) Casorati-Weierstrass

Consider the function $f(z) = e^{1/z}$. $f(z)$ has an essential (wesentlich) singularity at $z = 0$. Plot this function to see that in **any** neighborhood of $z = 0$, $f(z)$ is arbitrarily close to any complex number. That is: $\forall a \in \mathbb{C}, \forall \rho > 0$, and $\forall \epsilon > 0$, $\exists z_1$ with $|z_1| < \rho$ and $|f(z_1) - a| < \epsilon$.

Now consider the function $f(z) = 1/z^p$ which has a pole at $z = 0$. Show that in any neighborhood of $z = 0$ $f(z)$ is not arbitrarily close to any complex number.

This is an example of the Casorati-Weierstrass theorem: In any neighborhood of an essential singularity, the function value is arbitrarily close to any complex number.

2) Residues.

a) Consider the Möbius transformation $g(w) = (aw + b)/(cw + d)$ with $ad - bc \neq 0$. Show that for a given complex function $f(z)$

$$\mathrm{Res}(f(z), z_0) = \mathrm{Res}\left(f(g(\omega))\frac{dg(\omega)}{d\omega}, \omega_0\right)$$

where $z_0 = g(\omega_0)$. Be sure to show that the orientation of the path around ω_0 is correct. With this transformation law for residues, relate the residue at $z = 0$ to that at $\omega_0 = \infty$.

b) Compute the residues around the point z_0 for the functions:

1. $f(z) = \dfrac{1}{(z - z_0)^n}$
2. $f(z) = e^{1/(z-z_0)}$
3. $f(z) = \dfrac{\sin(z - z_0)}{z - z_0}$
4. $f(z) = \dfrac{g(z)}{(z - z_0)^n}$

Here $n \in \mathbb{N}$ and $g(z)$ is analytical in the neighborhood of z_0.

c) Consider the polynomial $P_2(z) = az^2 + bz + c$ and compute all residues of the function $g_2(z) = 1/P_2(z)$. Show that the sum over all residues of $g_2(z)$ vanishes. Now consider $P_1(z) = az + b$. Show again that the sum over all residues of $g_1(z) = 1/P_1(z)$ vanishes.

3) Consider the function:

$$f(z) = \frac{1}{(z - z_0)^p (z - z_1)^q}$$

with $z_0 \neq z_1$ and $p, q \neq 0 \in \mathbb{N}$. Compute the Laurent series around the point z_0. Hint. If you use the Cauchy formula, $g^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{g(z)}{(z-z_0)^{n+1}} dz$ [$g(z)$ analytical in $C_r(z_0)$], you will not have to compute any integrals explicitly!