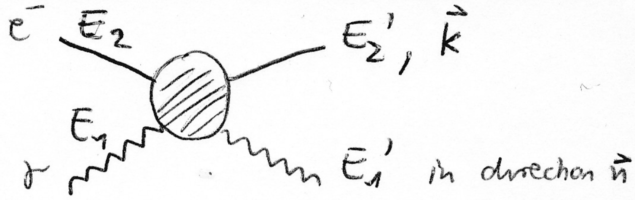
Sheet 1,



1, Momenta: $\vec{e}_z \frac{E_1}{c} = \vec{n} \frac{E_1'}{c} + \vec{k}$

Energy: $E_{el} = \sqrt{m^2c^4 + \vec{k}^2 c^2}$

$$E_1 + mc^2 = E_1' + \sqrt{m^2c^4 + \vec{k}^2c^2}$$

$$\Leftrightarrow m^2c^4 + \vec{k}^2c^2 = \left[E_1 - E_1' + mc^2\right]^2$$

$$\Leftrightarrow m^2c^4 + \left[\vec{e}_z E_1 - \vec{n} E_1'\right]^2 = E_1^2 + E_1'^2 + m^2c^4 - 2E_1E_1' + 2(E_1 - E_1')mc^2$$

$$\Leftrightarrow E_1^2 + E_1'^2 - 2\underset{\vec{e}_z \circ \vec{n}}{\underset{\shortparallel}{\cos\theta}}\, E_1 E_1' = E_1^2 + E_1'^2 - 2E_1E_1' + 2(E_1 - E_1')mc^2$$

$$\Leftrightarrow \cos\theta = 1 - \frac{E_1 - E_1'}{E_1 E_1'} mc^2 \quad , E = \frac{\hbar c}{\lambda}$$

$$\Leftrightarrow 1 - \cos\theta = \frac{mc}{\hbar}(\lambda' - \lambda) \quad \Leftrightarrow \Delta\lambda = \frac{\hbar}{mc}(1 - \cos\theta)$$

2, a, $i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi \Leftrightarrow \dot{\psi} = \frac{i\hbar}{2m}\Delta\psi + \frac{V}{i\hbar}\psi$

$$\Leftrightarrow \frac{i\hbar}{2m}\Delta\psi = \dot{\psi} - \frac{V}{i\hbar}\psi$$

$$\frac{i\hbar}{2m}\Delta\psi^* = -\dot{\psi}^* - \frac{V}{i\hbar}\psi^*$$

$$\nabla_n j_n = \frac{\hbar i}{2m}\left[\nabla_n(\nabla_n\psi^* \cdot \psi) - \nabla_n(\psi^* \nabla_n \psi)\right]$$

$$= \frac{\hbar i}{2m}\left[\psi\Delta\psi^* + \nabla_n\psi^*\nabla_n\psi - \nabla_n\psi^*\nabla_n\psi - \psi^*\Delta\psi\right]$$

$$= -\psi\dot{\psi}^* - \frac{V}{i\hbar}\psi\psi^* - \psi^*\dot{\psi} + \frac{V}{i\hbar}\psi\psi^* = -\frac{\partial}{\partial t}(\psi^*\psi)$$

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2b,
$$\frac{d}{dt}\int d^3x\ |\Psi|^2 = \int d^3x\ (\Psi^*\dot{\Psi} + \dot{\Psi}^*\Psi) = -\int d^3x\ \vec{\nabla}\circ\vec{j}$$

$$= -\lim_{R\to\infty}\int_{S(R)} \vec{j}\,\vec{n}\,dA$$

We have to impose that the current $\vec{j}$ decays faster than $\frac{1}{|\vec{x}|^2}$ since the surface grows with R^2

c,
$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi + \alpha$$

$$\Rightarrow \dot{\Psi} = \frac{i\hbar}{2m}\Delta\Psi + \frac{V\Psi}{i\hbar} + \frac{\alpha}{i\hbar} \qquad \dot{\Psi}^* = -\frac{i\hbar}{2m}\Delta\Psi^* - \frac{V}{i\hbar}\Psi^* - \frac{\alpha^*}{i\hbar}$$

$$\frac{\partial}{\partial t}|\Psi|^2 = \dot{\Psi}^*\Psi + \Psi^*\dot{\Psi}$$
$$= -\frac{i\hbar}{2m}\Delta\Psi^*\cdot\Psi - \frac{V}{i\hbar}\Psi^*\Psi - \frac{\alpha}{i\hbar}\Psi$$
$$+ \frac{i\hbar}{2m}\Delta\Psi\cdot\Psi^* + \frac{V}{i\hbar}\Psi^*\Psi + \frac{\alpha^*}{i\hbar}\Psi^*$$

$$= -\vec{\nabla}\circ\vec{j} + \frac{1}{i\hbar}(\alpha^*\Psi - \alpha\Psi^*)$$

If $\int \vec{\nabla}\circ\vec{j}\ d^3x = 0$, then $\int \frac{\partial}{\partial t}|\Psi|^2 d^3x \neq 0$

We have no conservation of $\int|\Psi|^2 d^3x$, so we cannot use it as a probability, or $|\Psi|^2$ as a probability density

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3a, $$\Psi(x,t=0) = \int \frac{d^3p}{(2\pi\hbar)^3}\, \tilde{\Psi}(p,t=0)\, e^{\frac{i}{\hbar}\vec{p}\vec{x}}$$

$$i\hbar \frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\int \frac{d^3p}{(2\pi\hbar)^3} \left(\frac{i}{\hbar}\right)^2 \vec{p}^{\,2}\, \tilde{\Psi}(p,t)\, e^{\frac{i}{\hbar}\vec{p}\vec{x}}$$

$$= \int \frac{d^3p}{(2\pi\hbar)^3}\, \frac{\vec{p}^{\,2}}{2m} \tilde{\Psi}(p,t)\, e^{\frac{i}{\hbar}\vec{p}\vec{x}}$$

$$\int \frac{d^3p}{(2\pi\hbar)^3}\, \dot{\tilde{\Psi}}(p,t) e^{\frac{i}{\hbar}\vec{p}\vec{x}} \stackrel{!}{=} \int \frac{d^3p}{(2\pi\hbar)^3}\, \frac{\vec{p}^{\,2}}{2mi\hbar}\, \tilde{\Psi}(p,t)\, e^{\frac{i}{\hbar}\vec{p}\vec{x}}$$

Lösung $\dot{f} = \frac{E_{\vec{p}}}{i\hbar} f \Rightarrow f(t) = e^{-\frac{iE_{\vec{p}}t}{\hbar}} f(t=0)$

$$\Rightarrow \Psi(x,t) = \int \frac{d^3p}{(2\pi\hbar)^3}\, \tilde{\Psi}(p,t=0)\, e^{\frac{i}{\hbar}(\vec{p}\vec{x}-E_{\vec{p}}t)} = \int \frac{d^3p}{(2\pi\hbar)^3}\, \tilde{\Psi}(p,t) e^{\frac{i}{\hbar}\vec{p}\vec{x}}$$

where $\tilde{\Psi}(p,t) = \tilde{\Psi}(p,0)\, e^{-\frac{i}{\hbar}E_{\vec{p}}t}$

b, $$\Psi^*(x,t)\Psi(x,t) = \int \frac{d^3p}{(2\pi\hbar)^3}\frac{d^3k}{(2\pi\hbar)^3}\, \tilde{\Psi}^*(p,t=0)\tilde{\Psi}(k,t=0) \otimes$$

$$\otimes\, e^{\frac{i}{\hbar}\left[(\vec{k}-\vec{p})\vec{x} - (E_{\vec{k}}-E_{\vec{p}})t\right]}$$

use $\int d^3x\; e^{\frac{i}{\hbar}\vec{l}\vec{x}} = (2\pi\hbar)^3\, \delta(\vec{l})$

$$\Rightarrow \int d^3x\; \Psi^*(x,t)\Psi(x,t) = \int \frac{d^3p}{(2\pi)^3\hbar^6}\, \tilde{\Psi}^*(p,t=0)\tilde{\Psi}(p,t=0)\,\hbar^3$$

$$= \int \frac{d^3p}{(2\pi\hbar)^3}\, \tilde{\Psi}^*(p,t=0)\,\tilde{\Psi}(p,t=0) = \int \frac{d^3p}{(2\pi\hbar)^3}\, \tilde{\Psi}^*(p,t)\tilde{\Psi}(p,t)$$

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3 c,

$$\psi^*(x,t)\, i\hbar \frac{\partial}{\partial t}\psi(x,t) = i\hbar \int \frac{d^3p}{(2\pi\hbar)^3}\frac{d^3k}{(2\pi\hbar)^3}\tilde{\psi}^*(p,0)\tilde{\psi}(k,0)\frac{-i}{\hbar}E_k \otimes$$

$$\otimes\; e^{\frac{i}{\hbar}\left[(\vec{k}-\vec{p})\vec{x}-(E_k-E_p)t\right]}$$

$$\int d^3x\, \psi^*\, i\hbar\frac{\partial}{\partial t}\psi = \int\frac{d^3p}{(2\pi\hbar)^3}\;\tilde{\psi}^*(p,t=0)\psi(p,t=0)\;E_{\vec{p}}$$

$$= \int\frac{d^3p}{(2\pi\hbar)^3}\;|\tilde{\psi}(p,t)|^2\, E_{\vec{p}}\,, \qquad E_{\vec{p}} = \frac{\vec{p}^{\,2}}{2m}$$

3d,
$$\langle x_i\rangle = \int\frac{d^3p}{(2\pi\hbar)^3}\,\tilde{\psi}(\vec{p},t)^*\; i\hbar\frac{\partial}{\partial p_i}\tilde{\psi}(\vec{p},t)$$

$$\tilde{\psi}(\vec{p},t) = \int d^3x\;\psi(\vec{x},t)\,e^{-\frac{i}{\hbar}\vec{p}\vec{x}}$$

$$i\hbar\frac{\partial}{\partial p_i}\tilde{\psi}(\vec{p},t) = i\hbar\int d^3x\;\psi(\vec{x},t)\,e^{-\frac{i}{\hbar}\vec{p}\vec{x}}\left(-\frac{i}{\hbar}x_i\right)$$

$$\Rightarrow \int\frac{d^3p}{(2\pi\hbar)^3}\tilde{\psi}^*(p,t)\; i\hbar\frac{\partial}{\partial p_i}\tilde{\psi}(p,t) = \int\frac{d^3p}{(2\pi\hbar)^3}\int d^3x\; d^3y\;\psi^*(\vec{x},t)\,y_i\,\psi(\vec{y},t)\,e^{-\frac{i}{\hbar}\vec{p}(\vec{y}-\vec{x})}$$

$$= \int d^3x\; d^3y\;\psi^*(\vec{x},t)\,y_i\,\psi(\vec{y},t)\,\delta(\vec{y}-\vec{x})$$

$$= \int d^3x\;\psi^*(\vec{x},t)\,x_i\,\psi(\vec{x},t) \equiv \langle x_i\rangle$$