Corrections Sheet 11

1 a)

$$\begin{aligned}
[S_i^{(1)}+S_i^{(2)},S_j^{(1)}+S_j^{(2)}] &= [S_i^{(1)},S_j^{(1)}]+[S_i^{(2)},S_j^{(2)}]+[S_i^{(2)},S_j^{(1)}]+[S_i^{(1)},S_j^{(2)}] \\
&= i\hbar\epsilon_{ijk}S_k^{(1)}+i\hbar\epsilon_{ijk}S_k^{(2)} \\
&= i\hbar\epsilon_{ijk}S_k^{\text{tot}}
\end{aligned}$$

b)
To prove eq.5, consider Sheet 10 1b):

$$\begin{aligned}
& e^{ie(\hat S^{(1)}+\hat S^{(2)})\theta/\hbar}S_k^{(1)}S_k^{(2)}e^{-ie(\hat S^{(1)}+\hat S^{(2)})\theta/\hbar} \\
= \; & e^{ie\hat S^{(2)}\theta/\hbar}S_k^{(2)}e^{-ie\hat S^{(2)}\theta/\hbar}e^{ie\hat S^{(1)}\theta/\hbar}S_k^{(1)}e^{-ie\hat S^{(1)}\theta/\hbar} \\
\overset{10,\,1b)}{=} \; & R_{kl}(e,\theta)S_l^{(1)}\cdot R_{km}(e,\theta)S_m^{(2)}=S_k^{(1)}S_k^{(2)}
\end{aligned}$$

$$\begin{aligned}
\hat U^\dagger(e,\theta)\hat H\hat U(e,\theta) &= \hat H \Rightarrow \frac{\partial}{\partial\theta}\hat U^\dagger\hat H\hat U=0 \\
&\Rightarrow \frac{\partial}{\partial\theta}\left(e^{ie\hat S\theta/\hbar}\hat He^{-ie\hat S\theta/\hbar}\right)=0 \\
&\Rightarrow e^{ie\hat S\theta/\hbar}\left(ie\hat S\hat H-i\hat He\hat S\right)e^{-ie\hat S\theta/\hbar}=0 \\
\text{(e was arbitrary)} &\Rightarrow [\hat S_i,\hat H]=0
\end{aligned}$$

c)
First, what is $S_i^{(1)}S_i^{(2)}$? First note that

$$S_i^{(1)}S_i^{(2)}=S_3^{(1)}S_3^{(2)}+\frac{1}{4}(S_+^{(1)}+S_-^{(2)})(S_+^{(1)}+S_-^{(2)})-\frac{1}{4}(S_-^{(1)}-S_+^{(2)})(S_-^{(1)}-S_+^{(2)})$$

where $S_+|\downarrow\rangle=\hbar|\uparrow\rangle$, $S_-|\uparrow\rangle=\hbar|\downarrow\rangle$, so we get

$$\begin{aligned}
S_i^{(1)}S_i^{(2)}|\uparrow\rangle|\uparrow\rangle &= \frac{\hbar^2}{4}|\uparrow\rangle|\uparrow\rangle+\frac{\hbar^2}{4}|\downarrow\rangle|\downarrow\rangle-\frac{\hbar^2}{4}|\downarrow\rangle|\downarrow\rangle=\frac{\hbar^2}{4}|\uparrow\rangle|\uparrow\rangle \\
S_i^{(1)}S_i^{(2)}|\downarrow\rangle|\downarrow\rangle &= \frac{\hbar^2}{4}|\downarrow\rangle|\downarrow\rangle+\frac{\hbar^2}{4}|\uparrow\rangle|\uparrow\rangle-\frac{\hbar^2}{4}|\uparrow\rangle|\uparrow\rangle=\frac{\hbar^2}{4}|\downarrow\rangle|\downarrow\rangle \\
&\text{And} \\
S_i^{(1)}S_i^{(2)}|\downarrow\rangle|\uparrow\rangle &= -\frac{\hbar^2}{4}|\downarrow\rangle|\uparrow\rangle+\frac{\hbar^2}{4}|\uparrow\rangle|\downarrow\rangle+\frac{\hbar^2}{4}|\uparrow\rangle|\downarrow\rangle \\
S_i^{(1)}S_i^{(2)}|\uparrow\rangle|\downarrow\rangle &= -\frac{\hbar^2}{4}|\uparrow\rangle|\downarrow\rangle+\frac{\hbar^2}{4}|\downarrow\rangle|\uparrow\rangle+\frac{\hbar^2}{4}|\downarrow\rangle|\uparrow\rangle \\
&\Downarrow \\
S_i^{(1)}S_i^{(2)}(|\uparrow\rangle|\downarrow\rangle+|\downarrow\rangle|\uparrow\rangle) &= \frac{\hbar^2}{4}(|\uparrow\rangle|\downarrow\rangle+|\downarrow\rangle|\uparrow\rangle) \\
S_i^{(1)}S_i^{(2)}(|\uparrow\rangle|\downarrow\rangle-|\downarrow\rangle|\uparrow\rangle) &= -\frac{3\hbar^2}{4}(|\uparrow\rangle|\downarrow\rangle-|\downarrow\rangle|\uparrow\rangle)
\end{aligned}$$

Now I express the total spin in terms of the product above

$$(S_i^{tot})^2=(S_i^{(1)}+S_i^{(2)})^2 = S_i^{(1)}S_i^{(1)}+S_i^{(2)}S_i^{(2)}+2S_i^{(1)}S_i^{(2)}$$

$$\begin{aligned}
\Rightarrow S_i^{(1)}S_i^{(2)} &= \frac{1}{2}\left(S_i^{tot}S_i^{tot}-S_i^{(1)}S_i^{(1)}-S_i^{(2)}S_i^{(2)}\right) \\
&= \frac{1}{2}\left(S_i^{tot}S_i^{tot}-\frac{3\hbar^2}{4}-\frac{3\hbar^2}{4}\right)=\frac{1}{2}(S_i^{tot})^2-\frac{3\hbar^2}{4}
\end{aligned}$$

You can conclude now that $S^{tot2}|0,0\rangle = 0$, $S^{tot2}|1,m\rangle = 2\hbar^2|1,m\rangle$

So we have the Hamiltonian in terms of total Spin. The states given in (7) are eigenstates to $(S_i^{tot})^2$ and thus of $\hat{H} = J(1/2S^{tot2} - 3\hbar^2/4)$ as well.

$$
\begin{aligned}
\Rightarrow \quad \hat{H}|0,0\rangle &= -J\frac{3\hbar^2}{4}|0,0\rangle \\
\hat{H}|1,m\rangle &= J\frac{\hbar^2}{4}|1,m\rangle
\end{aligned}
$$

Determining the action of $S_3^{(1)} + S_3^{(2)}$ on the states is straightforward:

$$
\begin{aligned}
(S_3^{(1)} + S_3^{(2)})|\uparrow\rangle|\uparrow\rangle &= \frac{(\hbar+\hbar)}{2}|\uparrow\rangle|\uparrow\rangle \\
(S_3^{(1)} + S_3^{(2)})|\downarrow\rangle|\downarrow\rangle &= \frac{(-\hbar-\hbar)}{2}|\downarrow\rangle|\downarrow\rangle \\
(S_3^{(1)} + S_3^{(2)})(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle) &= \frac{(\hbar-\hbar)}{2}|\uparrow\rangle|\downarrow\rangle + \frac{(\hbar-\hbar)}{2}|\downarrow\rangle|\uparrow\rangle = 0 \\
(S_3^{(1)} + S_3^{(2)})(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) &= \frac{(\hbar-\hbar)}{2}|\uparrow\rangle|\downarrow\rangle - \frac{(\hbar-\hbar)}{2}|\downarrow\rangle|\uparrow\rangle = 0
\end{aligned}
$$

To determine whether the four states constitute a basis, note that we have $2 \otimes 2$ state combinations of the two single spins, which results in the $3 \oplus 1$ linearly independed states given in (7). We know that they are eigenstates to the hermitian operators S_3 and S^2 with different eigenvalues, thus they are mutually orthogonal, and knowing that the Hilbert space has dimension 4, they constitute a basis. You could check this by calculating $\sum_{n=1}^{4} |n\rangle\langle n| = 1_{\mathcal{H}} = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\uparrow\downarrow\rangle\langle\uparrow\downarrow|$

d)
To show the property, one can demonstrate that the $S_\pm$ commute with the Hamiltonian. The classical intuition is that the Hamiltonian depends on the relative orientation of the two spins only. Once this is chosen ($l = 1$), the energy cannot depend on the absolute direction of the total spin, i.e. is independent from m.
e)
By switching on the magnetic field in the z direction, rotational invariance around the x- and y-axes is broken. We can't expect that $S_\pm$ which are made up of $S_{1,2}$ still commute with the Hamiltonian ($U^\dagger HU \neq H$), and the energy will depend on m. There is still rotational symmetry about the z-axis and therefore eigenstates of $\hat{H}$ can still have a sharp m ($[S_{tot3}, H] = 0$). Furthermore, semi-classically we would expect precession of the angular momentum around the magnetic field, which does not change the absolute value of total angular momentum ($[S_{tot}^2, H] = 0$).

i.

$$U^\dagger HU \neq H \Leftrightarrow [H, U] \neq 0$$

This is the case since the derivative is nonzero

$$\frac{\partial}{\partial\theta}[H, U] \neq 0 \Leftrightarrow [H, \frac{\partial}{\partial\theta}U] \neq 0$$

for $e_1 \neq 0$ or $e_2 \neq 0$ because $[S_{1,2}, H] = \frac{g\mu B}{\hbar}[S_{1,2}, S_3] \neq 0$

ii. $[H, S_{tot}^2] \propto [S_{tot3}, S_{tot}^2]$ because the Hamiltonian without magnetic field $JS_i^{(1)}S_i^{(2)}$ commutes (eq.6). We know that the components of angular momentum commute with total angular momentum squared because they satisfy the usual algebra. Also, $[S_{tot3}, S_{tot3}] = 0$

iii. The states in (7) were eigenstates of $\hat{H}$ without magnetic field. They are eigenstates of S_{tot3} as well (part c). Thus they are eigenstates of the new Hamiltonian.