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Sheet 4

1, a. No matter which observable $\hat{A}$ we choose to measure, the global phase $e^{i\phi}$ commutes with it ($[\hat{A}, e^{i\phi}]=0$) and will cancel:

$$\langle \psi_1(t) | e^{-i\phi} \hat{A} e^{i\phi} | \psi_1(t) \rangle$$

$$= \langle \psi_1(t) | e^{-i\phi} e^{i\phi} \hat{A} | \psi_1(t) \rangle = \langle \psi_1(t) | \hat{A} | \psi_1(t) \rangle$$

$\Rightarrow$ not observable

b. The relative phase between the states in a superposition is measurable due to interference effects:

$$\left(\langle \psi_1(t) | + e^{-i\phi} \langle \psi_2(t) | \right) \hat{A} \left(|\psi_1(t)\rangle + e^{i\phi} |\psi_2(t)\rangle \right)$$

$$= \langle \psi_1(t) | \hat{A} | \psi_1(t) \rangle + \langle \psi_2(t) | \hat{A} | \psi_2(t) \rangle$$

$$\underbrace{+2\mathrm{Re}\left(e^{-i\phi} \langle \psi_2(t) | \hat{A} | \psi_1(t) \rangle \right)}_{\text{Interference dept. on } \phi}$$

c. There are several ways to answer this question. If we write ρ with respect to the basis $\{|\psi_1\rangle, |\psi_2\rangle\}$, we get

$$\rho = \begin{bmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{bmatrix}$$

This matrix is regular ($\rho v = 0 \Rightarrow v = 0$).
Any matrix of the form $\tilde{\rho} = |\psi\rangle\langle\psi|$, $|\psi\rangle = a|\psi_1\rangle + b|\psi_2\rangle$ is singular, since $\tilde{\rho}\left(b|\psi_1\rangle - a|\psi_2\rangle\right) = 0$, so there is no $|\psi\rangle$ such that $\tilde{\rho} = \rho$. The physical meaning of this is that ρ corresponds to a statistical mix of $|\psi_1\rangle$ and $|\psi_2\rangle$, not a quantum superposition.

Another way to see this is to use the ansatz

$$\cancel{\rho = (a^* \langle\psi_1| + b^* \langle\psi_2|}$$

$$\hat{\tilde{\rho}} = (a|\psi_1\rangle + b|\psi_2\rangle)(a^*\langle\psi_1| + b^*\langle\psi_2|)$$

which, in the basis $\{|\psi_1\rangle, |\psi_2\rangle\}$ is

$$\tilde{\rho} = \begin{bmatrix} aa^* & ab^* \\ a^*b & bb^* \end{bmatrix}$$

Here, the "Interference" ab^* and a^*b makes it impossible to choose a, b such that $\tilde{\rho} = \rho$

d) i) The expectation value is given by

$$\mathrm{tr}(\hat{O}\hat{\rho}) \underset{\substack{\uparrow \\ \text{(linearity)}}}{=} \frac{1}{2}\mathrm{tr}(|x\rangle\langle x|\psi_1\rangle\langle\psi_1|) + \frac{1}{2}\mathrm{tr}(|x\rangle\langle x|\psi_2\rangle\langle\psi_2|)$$

~~use $\mathrm{tr}(ab\ldots ef) = \mathrm{tr}(fab\ldots e)$ etc.~~ ..

$$\cancel{\tfrac{1}{2}\mathrm{tr}} = \frac{1}{2}\langle\psi_1|x\rangle\langle x|\psi_1\rangle\langle\psi_1|\psi_1\rangle + \frac{1}{2}\langle\psi_2|x\rangle\langle x|\psi_2\rangle\langle\psi_2|\psi_2\rangle$$

$$= \frac{1}{2}|\langle x|\psi_1\rangle|^2 + \frac{1}{2}|\langle x|\psi_2\rangle|^2$$

We see again that ρ describes a statistical mix: the summation takes place after squaring the amplitudes → **no interference**

ii)
$$\langle\psi|\hat{O}|\psi\rangle = \frac{1}{2}\langle\psi_1|\hat{O}|\psi_1\rangle + \frac{1}{2}\langle\psi_2|\hat{O}|\psi_2\rangle + \frac{1}{2}\langle\psi_1|\hat{O}|\psi_2\rangle + \frac{1}{2}\langle\psi_2|\hat{O}|\psi_1\rangle$$

$$= \frac{1}{2}|\langle\psi_1|x\rangle|^2 + \frac{1}{2}|\langle\psi_2|x\rangle|^2 + \underbrace{\mathrm{Re}(\langle x|\psi_1\rangle\langle\psi_2|x\rangle)}_{\text{Interference}}$$

③

2) This problem adresses the consequences when measuring noncommuting observables

a) The apparatus which measures $\vec{m}\hat{\vec{\sigma}}$ can be thought of like a Stern-Gerlach experiment which is tilted relative to the z-axis. The possible outcomes of the experiment are given by the eigenvalues of $\vec{m}\hat{\vec{\sigma}}$. (choose basis $\{|1\rangle, |2\rangle\}$)

$$\det\left(\vec{m}\vec{\sigma} - \lambda \mathbb{1}\right) = \det\left(\begin{bmatrix} m_z-\lambda & m_x - im_y \\ m_x + im_y & -m_z - \lambda \end{bmatrix}\right)$$

$$= \underbrace{-m_z^2 - m_x^2 - m_y^2}_{=1} + \lambda^2 \quad \Rightarrow \quad \lambda = \pm 1$$

b) The expectation value is $\langle 1|\vec{m}\hat{\vec{\sigma}}|1\rangle = m_z$

Since $m_z = \langle \vec{m}\hat{\vec{\sigma}}\rangle = (1)\cdot P_1 + (-1)P_{-1} = 1\cdot P_1 - (1-P_1)$

$$\Rightarrow P_1 = \frac{m_z+1}{2} \qquad P_{-1} = \frac{1-m_z}{2}$$

c) i) This is equivalent to preparing initial states $|1\rangle$

ii) This will happen with probability $P_1 = \frac{m_z+1}{2}$

To determine the state $|1_m\rangle$ the atoms are in after they pass, we have to calculate the eigenvector of $\vec{m}\vec{\sigma}$ w.r.t. the eigenvalue 1:

$$\begin{bmatrix} m_z & m_x - im_y \\ m_x + im_y & -m_z \end{bmatrix}\begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 1 \end{pmatrix} \Rightarrow a\, m_z + m_x - im_y = a$$

$$\Rightarrow a = \frac{m_x - im_y}{1-m_z}$$

After normalization we get

$$|1_m\rangle \simeq \frac{1}{\sqrt{1 + \frac{m_x^2+m_y^2}{(1-m_z)^2}}}\begin{pmatrix} \frac{m_x - im_y}{1-m_z} \\ 1 \end{pmatrix}$$

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iii) Now we measure in the z-direction again. If we get atoms in the state $|1_m\rangle$, the probability of measuring them with "spin down" in the z direction is given by $P'_{-1} = \langle 1_m | \left(|-1\rangle\langle -1| \right) |1_m\rangle = |\langle -1 | 1_m\rangle|^2$

$$\Rightarrow P'_{-1} = \left| (0,1) \begin{pmatrix} * \\ 1/\sqrt{1+\frac{m_x^2+m_y^2}{(1-m_z)^2}} \end{pmatrix} \right|^2 = \frac{1}{1+\frac{m_x^2+m_y^2}{(1-m_z)^2}}$$
$$= \frac{1-m_z}{2}$$

The total probability that an atom passes through the entire experiment is $P_{tot} = P_1 P'_{-1}$

$$= \frac{m_z+1}{2} \frac{1}{1+\frac{1-m_z^2}{(1-m_z)^2}} = \frac{m_z+1}{2} \frac{1}{1+\frac{1+m_z}{1-m_z}}$$

$$= \frac{1}{2} \frac{(m_z+1)(1-m_z)}{1+m_z+1-m_z} = \frac{1}{4}(1-m_z^2)$$

To maximize this, the second measurement has to be in the x-y plane, i.e. $m_z = 0$, perpendicular to the spin prepared in i)

To summarize, we have calculated $P_{tot} = |\langle 1 | 1_m\rangle|^2 |\langle 1_m | -1\rangle|^2$

Note that, without the intermediate measurement $\vec{m}\hat{\vec{\sigma}}$, the final probability would be zero!

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a) $[\Pi_x, \Pi_y] = [p_x - \frac{e}{c}A_x, p_y - \frac{e}{c}A_y]$

$= [p_x, -\frac{e}{c}A_y] + [-\frac{e}{c}A_x, p_y]$

$= -\frac{e}{c}\cdot(-i\hbar)\left([\partial_x, A_y] + [A_x, \partial_y]\right)$

$= \frac{ie\hbar}{c}\left(\partial_x A_y + A_y\partial_x - A_y\partial_x + A_x\partial_y - \partial_y A_x - A_x\partial_y\right)$

$= \frac{ie\hbar}{c}\left(\partial_x A_y - \partial_y A_x\right) = \frac{ie\hbar}{c}B$

b) $[\hat{a}, \hat{a}^+] = \frac{c}{2e\hbar B}[\Pi_x + i\Pi_y, \Pi_x - i\Pi_y] = \frac{c}{2e\hbar B}\Big([\Pi_x, -i\Pi_y] + [i\Pi_y, \Pi_x]\Big)$

$= \frac{c}{2e\hbar B}\cdot(-i)\cdot 2[\Pi_x, \Pi_y] = -i\frac{c}{e\hbar B}\cdot\frac{ie\hbar}{c}B = 1$

c) $a^+a = \frac{c}{2e\hbar B}\Big(\underbrace{\Pi_x^2 + \Pi_y^2}_{= 2m\hat{H} \text{ if } A_3 = 0} + i\Pi_x\Pi_y - i\Pi_y\Pi_x\Big)$

$= \frac{c}{\cancel{2}e\hbar B}\left(\cancel{2}m\hat{H} - \frac{e\hbar}{c}B\right) = \frac{mc}{\cancel{2}e\hbar B}\hat{H} - \frac{1}{2}$

$\Rightarrow \hat{H} = \underbrace{\frac{\cancel{2}e\hbar B}{mc}\left(a^+a + \frac{1}{2}\right)}_{\text{Harmonic oscillator with } \omega = \frac{eB}{mc}}$

$\Rightarrow E_n = \omega\hbar\left(n + \frac{1}{2}\right)$

I have assumed $A_3 = 0$. If a different gauge is chosen, we have to take into account $\hat{p}_z$ to get an gauge invariant Hamiltonian