Sheet 6

1a) $|k\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{ikn} |n\rangle \qquad k = \frac{2\pi m}{N}, \quad m = 1 \dots N$

i) $\langle k|k'\rangle = \frac{1}{N} \sum_{n=1}^{N}\sum_{n'=1}^{N} e^{i(k'n'-kn)} \langle n'|n\rangle = \frac{1}{N}\sum_{n'}\sum_{n} e^{i(k'n'-kn)} \delta_{n',n}$

$= \frac{1}{N}\sum_{n=1}^{N} e^{i(k-k')n}$

$k-k' = \frac{(m-m')2\pi}{N} \qquad = \frac{1}{N}\sum_{n=1}^{N} e^{\frac{i\Delta m\, 2\pi}{N} n} = \frac{1}{N}\sum_{n=1}^{N} \left(e^{\frac{i2\pi n}{N}}\right)^{\Delta m}$

$k=k' \Rightarrow \Delta m = 0, \quad \langle k|k\rangle = \frac{1}{N}\sum_{n=1}^{N} 1 = 1$

$k \neq k' \Rightarrow$ Trick: zeige $\langle k|k'\rangle = \underbrace{e^{\frac{i2\pi\Delta m}{N}}}_{\neq 1} \langle k|k'\rangle$,

$= e^{\frac{i2\pi\Delta m}{N}} \frac{1}{N}\sum_{n=1}^{N} e^{\frac{i2\pi n\Delta m}{N}} = \frac{1}{N}\sum_{n=1}^{N} e^{\frac{i2\pi(n+1)\Delta m}{N}} = \frac{1}{N}\sum_{n=2}^{N+1} e^{\frac{i2\pi n\Delta m}{N}}$

But: $e^{\frac{i2\pi\Delta m}{N}} = e^{\frac{i2\pi(N+1)\Delta m}{N}} \qquad = \frac{1}{N}\sum_{n=1}^{N} e^{\frac{i2\pi n\Delta m}{N}} = \langle k|k'\rangle$

$\Rightarrow \langle k|k'\rangle = 0$

ii) $|\psi\rangle = \sum_{n=1}^{N} a_n |n\rangle$ arbitrary state

$\left(\sum_k |k\rangle\langle k|\right)|\psi\rangle = \sum_k\sum_n a_n |k\rangle\langle k|n\rangle$

$= \frac{1}{\sqrt{N}} \sum_k\sum_n a_n |k\rangle\langle k|n\rangle = \frac{1}{\sqrt{N}}\sum_k\sum_n a_n |k\rangle e^{-ikn}$

$= \sum_n a_n \left(\frac{1}{\sqrt{N}}\sum_k e^{-ikn}|k\rangle\right)$

$\left(\text{but } \frac{1}{\sqrt{N}}\sum_k e^{-ikn} \frac{1}{\sqrt{N}}\sum_{m=1}^{N} e^{ikm}|m\rangle = \frac{1}{N}\sum_k\sum_m e^{ik(n-m)}|m\rangle = |n\rangle\right)$

$= \sum_n a_n |n\rangle = |\psi\rangle \qquad \Rightarrow \sum_k |k\rangle\langle k| = \mathbb{1}_{\mathcal{H}}$

1b) $\hat{T}|k\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{ikn} |n+1\rangle = \frac{1}{\sqrt{N}} \sum_{n=2}^{N+1} e^{ik(n-1)} |n\rangle$

$= \frac{1}{\sqrt{N}} e^{-ik} \sum_{n=2}^{N+1} e^{ikn} |n\rangle \underset{\uparrow}{=} e^{-ik} |k\rangle$

$e^{ik\cdot 1} = e^{ik(N+1)}, \; |N+1\rangle = |1\rangle$

c) It is $\hat{T} = \sum_{n=1}^{N} |n+1\rangle\langle n|$, we can write $\hat{H}$ as

$\Rightarrow \hat{H} = \epsilon_0 \sum_n |n\rangle\langle n| + J(\hat{T} + \hat{T}^\dagger)$

$\Rightarrow \hat{H}|k\rangle = \epsilon_0 |k\rangle + J(e^{-ik} + e^{ik})|k\rangle = [\epsilon_0 + 2J\cos(k)]|k\rangle$

$\Rightarrow$ $|k\rangle$ are eigenstates of $\hat{H}$ with eigenvalue $\epsilon_0 + 2\cos(k)$

$\Rightarrow \hat{H} = \sum_k \underbrace{(\epsilon_0 + 2\cos(k))}_{\equiv \epsilon(k)} |k\rangle\langle k|$

d) $|\psi(t=0)\rangle = |n_0\rangle$

$|n_0\rangle = \sum_k |k\rangle\langle k|n_0\rangle = \frac{1}{\sqrt{N}} \sum_k |k\rangle e^{-ikn_0}$

$|\psi(t)\rangle = e^{-i\hat{H}t}|n_0\rangle = \sum_k |k\rangle \frac{1}{\sqrt{N}} \exp[-i(kn_0 - \epsilon(k)t)]$

e) $\langle\psi(t)| = \sum_k \langle k| \frac{1}{\sqrt{N}} \exp[i(kn_0 - \epsilon(k)t)]$

$\Rightarrow \langle n|\psi(t)\rangle = \sum_k \frac{1}{\sqrt{N}} e^{ikn} \frac{1}{\sqrt{N}} \exp[-i(kn_0 - \epsilon(k)t)]$

$= \sum_k \frac{1}{N} \exp[-i(k(n_0 - n) - \epsilon(k)t)]$

$\langle\psi(t)|n\rangle\langle n|\psi(t)\rangle = |\langle n|\psi(t)\rangle|^2$

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2) $\hat{L} = \hat{X} \times \hat{P}$

$$\hat{L}_i = \epsilon_{ijk} \hat{X}_j \hat{P}_k = -i\hbar \epsilon_{ijk} \hat{X}_j \hat{\partial}_k$$

i) $[L_i, L_j] = -\hbar^2 \epsilon_{iab} \epsilon_{jcd} [\hat{X}_a \hat{\partial}_b, \hat{X}_c \hat{\partial}_d]$

$$= -\hbar^2 \epsilon_{iab} \epsilon_{jcd} \left(\hat{X}_a \hat{\partial}_b \hat{X}_c \hat{\partial}_d - (ab \leftrightarrow cd) \right)$$

$$= -\hbar^2 \epsilon_{iab} \epsilon_{jcd} \left(\hat{X}_a \delta_{bc} \hat{\partial}_d + \hat{X}_a \hat{X}_c \hat{\partial}_b \hat{\partial}_d - \hat{X}_c \delta_{da} \hat{\partial}_b - \hat{X}_a \hat{X}_c \hat{\partial}_b \hat{\partial}_d \right)$$

$$= -\hbar^2 \left(\epsilon_{iak} \epsilon_{jkd} \hat{X}_a \hat{\partial}_d - \epsilon_{ikb} \epsilon_{jck} \hat{X}_c \hat{\partial}_b \right)$$

$$= -\hbar^2 \left(\delta_{id} \delta_{aj} X_a \partial_d - \delta_{ij} \delta_{ad} X_a \partial_d - \delta_{bj} \delta_{ic} X_c \partial_b + \delta_{bc} \delta_{ij} X_c \partial_b \right)$$

$$= -\hbar^2 \left(\hat{X}_j \hat{\partial}_i - \hat{X}_i \hat{\partial}_j \right)$$

compare: $i\hbar \epsilon_{ijk} \hat{L}_k = i\hbar \epsilon_{ijk} \left(-i\hbar \epsilon_{kab} \hat{X}_a \hat{\partial}_b \right)$

$$= \hbar^2 \epsilon_{kij} \epsilon_{kab} \hat{X}_a \hat{\partial}_b = \hbar^2 \left(\hat{X}_i \hat{\partial}_j - \hat{X}_j \partial_i \right)$$

$$= [L_i, L_j]$$

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ii) $(\hat{L} \times \hat{L})_i = -\hbar^2 \epsilon_{ijk} \epsilon_{jab} \epsilon_{kcd} \hat{X}_a \hat{\partial}_b \hat{X}_c \hat{\partial}_d = -\hbar^2 (\epsilon_{jki} \epsilon_{jab}) \epsilon_{kcd}$ @

$$= -\hbar^2 \left(\delta_{ka} \delta_{ib} - \delta_{kb} \delta_{ia} \right) \epsilon_{kcd} \text{ @}$$

$$= -\hbar^2 \left(\epsilon_{acd} \delta_{ib} - \epsilon_{bcd} \delta_{ia} \right) \left(\hat{X}_a \delta_{bc} \hat{\partial}_d + \hat{X}_a \hat{X}_c \hat{\partial}_b \hat{\partial}_d \right)$$

$$= -\hbar^2 \left(\epsilon_{aid} \hat{X}_a \hat{\partial}_d + \underbrace{\epsilon_{acd} \hat{X}_a \hat{X}_c \hat{\partial}_i \hat{\partial}_d}_{=0} - \underbrace{\epsilon_{bcd} \hat{X}_i \hat{X}_c \hat{\partial}_b \hat{\partial}_d}_{=0} \right) \underbrace{- \hbar^2 \epsilon_{bcd} \delta_{bc} \cdots}_{=0}$$

$$= -\hbar^2 \epsilon_{ida} \hat{X}_a \hat{\partial}_d = \hbar^2 \epsilon_{ijk} \hat{X}_j \hat{\partial}_k = i\hbar \hat{L}_i$$

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and so on...