Sheet 7

Before we start: What is $L^+|m\rangle$?

$$L_3 L^+|m\rangle = L_3(L_1+iL_2)|m\rangle = L_3L_1|m\rangle + iL_3L_2|m\rangle$$

$$= m\hbar L_1|m\rangle + im\hbar L_2|m\rangle + [L_3,L_1]|m\rangle + i[L_3,L_2]|m\rangle$$

$$= m\hbar L^+|m\rangle + i\hbar L_2|m\rangle + \hbar L_1|m\rangle$$

$$= (m+1)\hbar\, L^+|m\rangle$$

$$\Rightarrow L^+|m\rangle \propto |m+1\rangle$$

Normalization:

$$\langle m|L^- L^+|m\rangle = \langle m|L_1^2+L_2^2|m\rangle = \langle m|\vec{L}^2|m\rangle - \langle m|L_3^2|m\rangle$$

$$= l(l+1) - |m| \qquad (*)$$

1a) Scattering of a wave coming from the left, at a δ-shaped peak (repulsive)

b) $\psi_I(0) = \psi_{II}(0)$ $\quad \psi'_{II}(0) - \psi'_I(0) = \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} -\frac{2m}{\hbar^2}(E-\delta(x)V_0)\psi(x)\,dx$

$$= \frac{2m}{\hbar^2}V_0\psi(0) = \frac{2m}{\hbar^2}V_0 S$$

c) $e^0 + Re^0 = Se^0 \Rightarrow 1+R=S$

$$iqS - iq + iqR = \frac{2m}{\hbar^2}V_0S \Rightarrow iq(S+R-1) = \frac{2m}{\hbar^2}V_0 S$$

$$\Rightarrow iq(2S-2) = \frac{2m}{\hbar^2}V_0S \Rightarrow +iq(S-1) = \frac{m}{\hbar^2}V_0\cdot S$$

$$\Rightarrow -iq = \left(\frac{m}{\hbar^2}V_0 - iq\right)S \Rightarrow S = \frac{iq}{\left(iq - \frac{mV_0}{\hbar^2}\right)}$$

d) $j_{x<0} = -\frac{\hbar q}{m}(R^*R-1) = \frac{\hbar q}{m}(1-R^*R)$

$$j_{x>0} = \frac{\hbar q}{m}S^*S$$

Note that $j_{x<0} = j_{x>0}$ because $\underline{2S^*S = S^*+S}$. This is a simple Version of the optical theorem.

2a) $\psi = K\,(x+y+2z)\,e^{-\alpha r}$

$= K(r\sin\theta\cos\varphi + r\sin\theta\sin\varphi + 2r\cos\theta)e^{-\alpha r}$

$= (s\theta\, c\varphi + s\theta\, s\varphi + 2c\theta)\, K r e^{-\alpha r}$

$c\theta = \sqrt{\frac{4\pi}{3}}\, Y_{1,0}$ $\quad s\theta s\varphi = \sqrt{\frac{2\pi}{3}}\, i\,(Y_{1,1} + Y_{1,-1})$

$s\theta c\varphi = \sqrt{\frac{2\pi}{3}}\,(-Y_{1,1} + Y_{1,-1})$

$$\Rightarrow \psi(r,\theta,\varphi) = \left[\sqrt{\frac{2\pi}{3}}(-Y_{1,1}+Y_{1,-1}) + \sqrt{\frac{2\pi}{3}}\, i\,(Y_{1,1}+Y_{1,-1}) + 2\sqrt{\frac{4\pi}{3}}\, Y_{1,0}\right] K r e^{-\alpha r}$$

b) $\frac{1}{\hbar} L_z \psi = \left[\sqrt{\frac{2\pi}{3}}\,(-Y_{1,1} - Y_{1,-1}) + \sqrt{\frac{2\pi}{3}}\, i\,(Y_{1,1} - Y_{1,-1})\right] K r e^{-\alpha r}$

$\psi^* = \left[\sqrt{\frac{2\pi}{3}}(-Y_{1,1}^* + Y_{1,-1}^*) - \sqrt{\frac{2\pi}{3}}\, i\,(Y_{1,1}^* + Y_{1,-1}^*) + 2\sqrt{\frac{4\pi}{3}}\, Y_{1,0}^*\right]$

$$\int d\Omega\, \psi^* \frac{L_z}{\hbar}\psi = K^2 r^2 e^{-2\alpha r}\left[\left(-\sqrt{\frac{2\pi}{3}} + i\sqrt{\frac{2\pi}{3}}\right)\left(-\sqrt{\frac{2\pi}{3}} - i\sqrt{\frac{2\pi}{3}}\right) + \left(-\sqrt{\frac{2\pi}{3}} - i\sqrt{\frac{2\pi}{3}}\right)\left(\sqrt{\frac{2\pi}{3}} - i\sqrt{\frac{2\pi}{3}}\right)\right]$$

$$= K^2 r^2 e^{-2\alpha r}\left[\frac{2\pi}{3} + \frac{2\pi}{3} - \frac{2\pi}{3} - \frac{2\pi}{3}\right] = 0$$

$\Rightarrow \langle\psi|L_z|\psi\rangle = 0$

also, $\hat{L}^2\, Y_{1,*} = \hbar^2\, 2\, Y_{1,*} = \hbar^2 (l+1) l\, Y_{1,*}$

there are only harmonics with $l=1$ in the wave

$\Rightarrow \langle\psi|\frac{\hat{L}^2}{\hbar^2}|\psi\rangle = 2$

3a) The Hamiltonian commutes with total angular momentum:

$$[\hat{L}^2, \hat{H}] = [\hat{L}^2, g\hat{L}^2] + [\hat{L}^2, B\cdot\hat{L}] = 0 + 0 = 0$$

What is $\langle l_1, m_1 | \hat{H} | l_2, m_2 \rangle$?

Trick: Insert $\hat{L}^2$:

$$\langle l_1, m_1 | \hat{H} \frac{\hat{L}^2}{\hbar^2} | l_2, m_2 \rangle = (l_2+1) l_2 \cdot \langle l_1, m_1 | \hat{H} | l_2, m_2 \rangle$$

$$= \langle l_1, m_1 | \frac{\hat{L}^2}{\hbar^2} \hat{H} | l_2, m_2 \rangle = (l_1+1) l_1 \langle l_1, m_1 | \hat{H} | l_2, m_2 \rangle$$

So, if $l_1 \neq l_2$, we get $\langle l_1, m_1 | \hat{H} | l_2, m_2 \rangle = 0$

We see again that angular momentum is conserved

$$\Rightarrow \hat{H} = \sum_{l=0}^{\infty} \sum_{m,m'=-l}^{l} h^{(l)}_{m,m'} |l,m\rangle\langle l,m'|$$

b) Let $l=1$ $\quad \langle m_1 | \hat{H} | m_2 \rangle = 2g\hbar^2 + B_i \langle m_1 | \hat{L}_i | m_2 \rangle$

i) $\langle m_1 | L_3 | m_2 \rangle = \hbar m_2 \delta_{m_1, m_2}$

$L^+ = L_1 + iL_2 \qquad L^- = L_1 - iL_2$

ii) $\langle m_1 | L_1 | m_2 \rangle = \frac{1}{2} \langle m_1 | L_+ | m_2 \rangle + \frac{1}{2} \langle m_1 | L_- | m_2 \rangle$

$$\overset{(*)}{=} \hbar \frac{1}{2}\sqrt{2}\, \delta_{m_1, m_2+1} + \hbar \frac{1}{2}\sqrt{2}\, \delta_{m_1, m_2-1} = \hbar \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\langle m_1 | L_2 | m_2 \rangle = -\frac{i}{2} \langle m_1 | L^+ - L^- | m_2 \rangle = -\frac{i}{2} \langle m_1 | L^+ | m_2 \rangle + \frac{i}{2} \langle m_1 | L^- | m_2 \rangle$$

$$= -\frac{i\hbar}{\sqrt{2}} \delta_{m_1, m_2+1} + \frac{i\hbar}{\sqrt{2}} \delta_{m_1, m_2-1}$$

$$= -\frac{i\hbar}{\sqrt{2}} \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{array}{l} m_1=1 \\ m_1=0 \\ m_1=-1 \end{array} \\ \begin{array}{ccc} m_2=1 & m_2=0 & m_2=-1 \end{array} \end{array}$$

$[K_x, K_y] = i\hbar K_z$

$[K_y, K_z] = i\hbar K_x$

$[K_z, K_x] = i\hbar K_y$

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3c) In the Heisenberg picture,

$$\hat{L}(t) = e^{i\frac{\hat{H}t}{\hbar}} \hat{L}(0) e^{-i\hat{H}\frac{t}{\hbar}}$$

$$\frac{d}{dt}\hat{L}_i(t) = \frac{i}{\hbar} e^{iHt/\hbar} [\hat{H}, \hat{L}_i] e^{-iHt/\hbar} = \frac{i}{\hbar} [\hat{H}, \hat{L}_i(t)]$$

$$[H, \hat{L}_i] = B_j [\hat{L}_j, \hat{L}_i] = -i\hbar \epsilon_{ijk} B_j \hat{L}_k = -i\hbar \vec{B}\times\hat{L}$$

$$\Rightarrow \frac{d}{dt}\hat{L}_i(t) = e^{iHt/\hbar} (\vec{B}\times\hat{L}) e^{-iHt/\hbar} = \vec{B}\times\vec{L}(t)$$

d) $\vec{B} = (B,0,0)$ $\qquad |\Psi(t)=0\rangle = |l, m=1\rangle$

$$\langle\hat{L}\rangle(t) = \langle l=1, m=1 | L(t) | l=1, m=1\rangle$$

$$\frac{d}{dt}L(t) = \begin{pmatrix} B \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} 0 \\ -B\cdot L_z(t) \\ B\cdot L_y(t) \end{pmatrix}$$

$$\begin{pmatrix} \langle \dot{L_x}\rangle \\ \langle \dot{L_y}\rangle \\ \langle \dot{L_z}\rangle \end{pmatrix} = B \begin{pmatrix} 0 \\ -\langle L_z\rangle \\ \langle L_y\rangle \end{pmatrix} \qquad \langle L_y\rangle(0) = 0 \quad \langle L_z\rangle(0) = 0$$

$$\Rightarrow \langle L_y\rangle(t) = \sin(-Bt)$$
$$\langle L_z\rangle(t) = \cos(-Bt)$$

we see a precession around the B-field as an axis