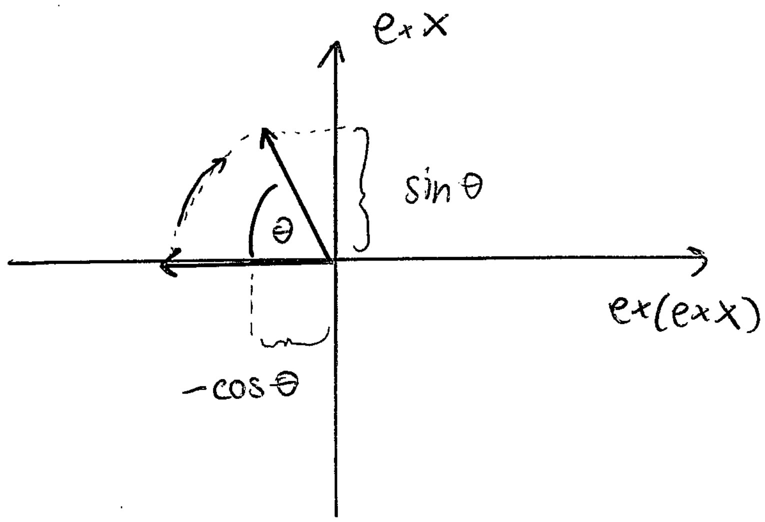
Sheet 8: Symmetries, Generators and Conservation Laws

1a)



b)

$$\frac{d}{d\theta} x(\theta) = \sin\theta\,(e\times(e\times x)) + \cos\theta\,(e\times x)$$

$$e\times x(\theta) = -\cos\theta\,[e\times(e\times(e\times x))] + \sin\theta\; e\times(e\times x)$$

$$a\times(b\times c) = b(ac) - c(ab)$$

$$e\times(e\times(e\times x)) = e(e(e\times x)) - (e\times x)(ee) = -e\times x$$

$$\Rightarrow\; e\times x(\theta) = \cos\theta\,(e\times x) + \sin\theta\,(e\times(e\times x)) = \frac{d}{d\theta}x(\theta) \quad \#$$

c)

$$\hat{p}(\theta) = e^{i\vec{e}\circ\hat{L}\frac{\theta}{\hbar}}\,\hat{p}\,e^{-i\vec{e}\circ\hat{L}\frac{\theta}{\hbar}}$$

$$\frac{d}{d\theta}\hat{p}_j(\theta) = e^{i@}\left(\tfrac{i}{\hbar}\,\vec{e}\circ\hat{L}\right)\hat{p}_j e^{-i@} + e^{i@}\hat{p}_j\left(-\tfrac{i}{\hbar}\,\vec{e}\circ\hat{L}\right)e^{-i@}$$

$$= \frac{i}{\hbar}\,e^{i@}\left[e\circ\hat{L},\hat{p}_j\right]e^{-i@}$$

$$[\hat{L}_i,\hat{p}_j] = i\hbar\,\epsilon_{ijk}\hat{p}_k$$

$$= -e^{i@}\; e_i p_k \epsilon_{ijk}\; e^{-i@} = \left[\vec{e}\times\hat{p}(\theta)\right]_j$$

d) $T_{e,\theta} = e^{-i\vec{e}\cdot\hat{L}\frac{\theta}{\hbar}}$

$$T_{\vec{e},\theta} T^{\dagger}_{\vec{e},\theta} = e^{-i\vec{e}\cdot\hat{L}\frac{\theta}{\hbar}} e^{i\vec{e}\cdot\hat{L}\frac{\theta}{\hbar}} = \mathbb{1} \quad \text{, insb. } T_{\vec{e},-\theta} = T^{\dagger}_{\vec{e},\theta} = T^{-1}_{\vec{e},\theta}$$

<u>Trick</u>

$$T^{\dagger}_{\vec{e}\theta} \hat{P} T_{\vec{e}\theta} |\vec{p}\rangle = \Big(R(e,\theta)\overset{\text{Operator}}{\hat{\vec{p}}}\Big) |\vec{p}\rangle \underset{\uparrow}{=} \Big(R(e,\theta)\overset{\text{Number}}{\vec{p}}\Big) |\vec{p}\rangle$$

now we can substitute the eigenvalue $\vec{p}$

$$\Rightarrow \hat{p}\left(T_{\vec{e},\theta} |\vec{p}\rangle\right) = T_{\vec{e},\theta} \left(R(e,\theta)\hat{p}\right) |\vec{p}\rangle$$

$$= T_{\vec{e},\theta} \left(R(e,\theta)\vec{p}\right) |\vec{p}\rangle = \left(R(e,\theta)\vec{p}\right)\left(T_{\vec{e}\theta} |\vec{p}\rangle\right)$$

acts on

$$\Rightarrow \left[\left(T_{\vec{e}\theta} |\vec{p}\rangle\right) \propto |R(e,\theta)\vec{p}\rangle\right]$$

Short digression: Conservation Laws in QM and Classical Mechanics

Classical

Dynamics: $H(q_i, p_i)$, $\dot{q}_i = \frac{\partial H}{\partial p_i}$ $-\dot{p}_i = \frac{\partial H}{\partial q_i}$

Observables: $f(q,p)$

$$\frac{d}{dt} f(q,p) = \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i = \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \equiv \{f, H\}$$

"Poisson bracket"

Conserved quantities: f conserved $\Leftrightarrow$ $\{f, H\} = 0$

Quantum

Dynamics: $\hat{H}(\hat{q}_i, \hat{p}_i)$

Observables: $\hat{f}(\hat{q}_i, \hat{p}_i)$

$$\frac{d}{dt}\hat{f}(\hat{q},\hat{p}) = -\frac{i}{\hbar}[\hat{f}, \hat{H}]$$

Conserved quantities: $\hat{f}$ conserved $\Leftrightarrow$ $[\hat{f}, \hat{H}] = 0$

Taking angular momentum as an example, we will see that both in the classical and the quantum case, the conserved observables generate symmetry transformations of the Hamiltonian

General case: <u>Noether theorem</u>

Classical

Let H be invariant under rotations, i.e. $\frac{d}{d\theta} H(q(\theta), p(\theta)) = 0$

$$\Leftrightarrow \frac{\partial H}{\partial q_i}(\vec{e}\times\vec{q})_i + \frac{\partial H}{\partial p_i}(\vec{e}\times\vec{p})_i = 0 \Leftrightarrow \{H, \vec{e}\circ(\vec{q}\times\vec{p})\} = 0$$

$$\Leftrightarrow \{H, \vec{e}\cdot\vec{\ell}\} = 0$$

$\Rightarrow$ Angular Momentum in the $\vec{e}$-direction is conserved, $\vec{e}$ was arbitrary $\Rightarrow$ $\vec{\ell}$ is conserved

In the poisson bracket, $\vec{\ell}$ generates infinitesimal rotations of an observable

Quantum

Let $\hat{H}$ be invariant under rotations $\Leftrightarrow \frac{d}{d\theta}\left(e^{i\vec{e}\circ\hat{L}\frac{\theta}{\hbar}}\hat{H}e^{-i\vec{e}\circ\hat{L}\frac{\theta}{\hbar}}\right)=0$

$$\Leftrightarrow e^{i\vec{e}\circ\hat{L}}[\vec{e}\circ\hat{L},\hat{H}]e^{-i\vec{e}\circ\hat{L}\frac{\theta}{\hbar}}=0$$

$$\Leftrightarrow [\vec{e}\circ\hat{L},\hat{H}]=0 \quad (*)$$

Again, $\hat{L}$ generates the symmetry transformation, and $[\hat{L},\hat{H}]=0$ tells us that angular momentum is conserved.

Conclusion

[Generators of symmetries are conserved Observables]

Example: $\hat{H}=\frac{\hat{p}\circ\hat{p}}{2m}+V(\hat{x}) \qquad V(x)=V(|x|)$

$$T^{\dagger}_{\vec{e}\theta}\,\hat{p}\circ\hat{p}\,T_{\vec{e}\theta} = T^{\dagger}_{e\theta}\hat{p}_j T_{e\theta}T^{\dagger}_{e\theta}\hat{p}_j T_{e\theta}$$

$$= \left(R_{jl}(e,\theta)\hat{p}_l\right)\left(R_{jl'}(e,\theta)\hat{p}_{l'}\right) = \hat{p}^T \underbrace{R^T(e,\theta)R(e,\theta)}_{=\mathbb{1}}\hat{p} = \hat{p}^2$$

Potential: $V(x)=\sum a_n x^n$

$$T^{\dagger}_{\vec{e}\theta}V(\hat{x})T_{\vec{e}\theta} = \sum_{\substack{n=0\\j}}^{\infty} a_{nj}\left(T^{\dagger}_{e\theta}\hat{X}_j T_{e\theta}\right)^n = V(R(e,\theta)\hat{x}) = V(\hat{x})$$

$$\Rightarrow \frac{d}{d\theta}\left(e^{i\vec{e}\hat{L}\frac{\theta}{\hbar}}\hat{H}e^{-i\vec{e}\circ\hat{L}\frac{\theta}{\hbar}}\right)=0 \overset{(*)}{\Rightarrow} [\vec{e}\circ\hat{L},\hat{H}]=0$$

$$\Rightarrow \hat{L} \text{ is conserved}$$