QM1 Exercises. Sheet 1.

Corrections May 8-9

1) Compton scattering. (5 points)

a) Using four momentum conservation, show that when a photon (wavelength λ) scatters off an electron initially at rest it's wavelength changes as:

$$\lambda' - \lambda = 4\pi \lambda\!\!\!^{-}_c \sin^2(\Theta/2). \tag{1}$$

Here, Θ is the angle between the momenta of the incoming and scattered photons, and $\lambda\!\!\!^{-}_c = \hbar/m_e c$ the Compton wavelength of the electron. (Note that $\lambda\!\!\!^{-}_c = \lambda_c/2\pi$).

b) Compute the energy of the scattered photon in terms of the energy of the incoming one.

2) Continuity equation. (5 points)

a) Consider the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\boldsymbol{x},t) = -\frac{\hbar^2}{2m}\Delta\Psi(\boldsymbol{x},t) + V(\boldsymbol{x})\Psi(\boldsymbol{x},t) \tag{2}$$

describing a particle in an external potential $V(\boldsymbol{x})$. Show that the continuity equation

$$\begin{aligned} &\frac{\partial}{\partial t}|\Psi(\boldsymbol{x},t)|^2 + \boldsymbol{\nabla}\boldsymbol{j}(\boldsymbol{x},t) = 0 \ \text{ with} \\ &\boldsymbol{j}(\boldsymbol{x},t) = \frac{\hbar i}{2m}\left[\left(\boldsymbol{\nabla}\Psi(\boldsymbol{x},t)^\star\right)\Psi(\boldsymbol{x},t) - \Psi(\boldsymbol{x},t)^\star\left(\boldsymbol{\nabla}\Psi(\boldsymbol{x},t)\right)\right] \end{aligned} \tag{3}$$

holds.

b) Assume that the wave function is normalized to unity (i.e. $\int \mathrm{d}^3\boldsymbol{x}|\Psi(\boldsymbol{x},t)|^2 = 1$ so that $\lim_{\boldsymbol{x}\to\infty}\Psi(\boldsymbol{x},t) = 0$) show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{x}|\Psi(\boldsymbol{x},t)|^2 = 0 \tag{4}$$

c) Show that if the Schrödingder differential equation were not homogeneous, that is

$$i\hbar\frac{\partial}{\partial t}\Psi(\boldsymbol{x},t) = -\frac{\hbar^2}{2m}\Delta\Psi(\boldsymbol{x},t) + V(\boldsymbol{x})\Psi(\boldsymbol{x},t) + d, \tag{5}$$

the above results do not hold. Argue why this leads to problems with the statistical interpretation of the wave function.

3) Free particle (5 points)

Assume that at $t = 0$ a free particle ($V(\boldsymbol{x}) = 0$)is described by the wave function

$$\Psi(\boldsymbol{x}, t = 0) = \int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3} \tilde{\Psi}(\boldsymbol{p}, t = 0) e^{\frac{i}{\hbar}\boldsymbol{p}\,\boldsymbol{x}}. \tag{6}$$

a) Using the Schrödinger equation, compute $\Psi(\boldsymbol{x}, t)$ in terms of a superposition of plane waves, each plane wave describing a particle with momentum $\boldsymbol{p}$ and energy $E_{\boldsymbol{p}}$.

b) Show that:

$$\int d^3\boldsymbol{x} |\Psi(\boldsymbol{x}, t)|^2 = \int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3} |\tilde{\Psi}(\boldsymbol{p}, t)|^2 \tag{7}$$

c) $\frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}|\tilde{\Psi}(\boldsymbol{p}, t)|^2$ corresponds to the probability that, at time t, the momentum of the particle lies in a momentum space volume element $d^3\boldsymbol{p}$ centered around $\boldsymbol{p}$. Hence, the expectation value of the energy of the particle reads:

$$\langle E \rangle = \int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3} |\tilde{\Psi}(\boldsymbol{p}, t)|^2 \frac{\boldsymbol{p}^2}{2m}. \tag{8}$$

Show that

$$\langle E \rangle = \int d^3\boldsymbol{x} \Psi^\star(\boldsymbol{x}, t) i\hbar \frac{\partial}{\partial t} \Psi(\boldsymbol{x}, t) \tag{9}$$

Show that $\langle E \rangle$ is time-independent as should be the case since energy is conserved.

d) Show that

$$\langle \boldsymbol{x} \rangle = \int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3} \tilde{\Psi}(\boldsymbol{p}, t)^\star i\hbar \nabla_{\boldsymbol{p}} \tilde{\Psi}(\boldsymbol{p}, t) \tag{10}$$