QM1 Exercises. Sheet 10.

Corrections July 17-18

1) Pauli Spin Matrices. (5 Points)

In the $|\uparrow\rangle$, $|\downarrow\rangle$ basis the Spin 1/2 operator, $\hat{\boldsymbol{S}}$, is represented by the Pauli spin matrices: $\langle s|\hat{\boldsymbol{S}}|s'\rangle = \frac{\hbar}{2}\boldsymbol{\sigma}_{s,s'}$

a) Show that:

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{i,j}. \tag{1}$$

Use the above to show that:

$$e^{-i\theta\boldsymbol{e}\cdot\boldsymbol{S}/\hbar} = \begin{pmatrix} \cos(\theta/2) - i\sin(\theta/2)e_z, & -i\sin(\theta/2)(e_x - ie_y) \\ -i\sin(\theta/2)(e_x + ie_y), & \cos(\theta/2) + i\sin(\theta/2)e_z \end{pmatrix}. \tag{2}$$

Here $\boldsymbol{e}$ is a unit vector and $\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$.

b) Inspiring yourself from exercise sheet 8 show that:

$$e^{i\theta\boldsymbol{e}\cdot\boldsymbol{S}/\hbar}\boldsymbol{S}e^{-i\theta\boldsymbol{e}\cdot\boldsymbol{S}/\hbar} = R(\boldsymbol{e},\theta)\boldsymbol{S} \tag{3}$$

where $R(\boldsymbol{e},\theta)$ is an $SO(3)$ rotation through an angle θ around the axis $\boldsymbol{e}$. Use this relation to find the representation of the Spin 1/2 operator in the $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$, $|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$ basis. You can check your result by direct calculation of the spin operator in the $|\rightarrow\rangle, |\leftarrow\rangle$ basis.

2) Hydrogen Atom in a magnetic field. (5 Points)

A hydrogen atom is in the state $2p$ with $m = \hbar$ and spin $|\rightarrow\rangle$. At time $t = 0$ a strong magnetic field in the z-direction is switched on.

a) Compute the wave function as a function of time. (You can neglect the spin-orbit coupling as well as the diamagnetic term).

b) Compute the expectation values of $\hat{L}_x$ and $\hat{S}_x$ as a function of time.

3) Spin 1/2 particle in crossed magnetic and electric fields. (5 Points)

A spin 1/2 particle of mass m, charge q and magnetic moment, $\frac{gq}{2mc}\hat{\boldsymbol{S}}$ moves in spatially constant crossed magnetic and electric fields $\boldsymbol{B} = B_0\boldsymbol{e}_z$ and $\boldsymbol{E} = E_0\boldsymbol{e}_x$. Neglecting spin orbit coupling, compute the full energy spectrum. Hint. Here is a choice of gauges which may help you: $\boldsymbol{A} = B_0x\boldsymbol{e}_y$, $\Phi = -E_0x$.