QM1 Exercises. Sheet 2.

Corrections May 15-16

1) Gaussian wave packet. (10 points)

In class we discussed the physics of the propagation of a wave packet. Now we have to do the calculations!

Consider a free particle in one dimension which at time $t = 0$ is described by the wave function:

$$\Psi(x, t = 0) = \frac{1}{(2\pi D)^{1/4}} e^{\frac{i}{\hbar}p_0 x} e^{-x^2/4D} \tag{1}$$

a) Compute $\tilde{\Psi}(p, t = 0)$. Hint. You can use the Eq.

$$\int_{-\infty}^{\infty} \mathrm{d}x \; e^{i\alpha x} e^{-\beta x^2} = \sqrt{\frac{\pi}{\beta}} e^{-\alpha^2/4\beta} \tag{2}$$

b) Since $\Psi(x, t)$ satisfies the Schrödinger Eq.

$$i\hbar \frac{\partial}{\partial t} \Psi(\boldsymbol{x}, t) = -\frac{\hbar^2}{2m} \Delta \Psi(\boldsymbol{x}, t), \tag{3}$$

show that $\tilde{\Psi}(p, t)$ satisfies

$$i\hbar \frac{\partial}{\partial t} \tilde{\Psi}(p, t) = \frac{p^2}{2m} \tilde{\Psi}(p, t) \tag{4}$$

which is nothing but the Schrödinger Eq. in momentum space. Compute $\tilde{\Psi}(p, t)$.

c) Show that:

$$\Psi(x, t) = \frac{1}{(2\pi D)^{1/4}} \left(\frac{D}{D(t)}\right)^{1/2} e^{\frac{i}{\hbar}(p_0 x - E(p_0)t)} e^{-(x - v_0 t)^2/4D(t)} \tag{5}$$

Here $E(p_0) = p_0^2/2m$, $v_0 = p_0/m$ and $D(t) = D + it\hbar/2m$

d) Show that

$$\langle p \rangle = p_0, \qquad \langle x \rangle = v_0 t$$
$$(\Delta p)^2 \equiv \langle (p - \langle p \rangle)^2 \rangle = \frac{\hbar^2}{4D}, \qquad (\Delta x)^2 \equiv \langle (x - \langle x \rangle)^2 \rangle = D\left(1 + \frac{\hbar^2 t^2}{4m^2 D^2}\right) \tag{6}$$

2) Zero point motion. (5 points)

The Schrödinger Eq. of the one-dimensional harmonic oscillator reads:

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \underbrace{\left[\frac{\hat{P}^2}{2m} + \frac{m\omega^2}{2} \hat{X}^2\right]}_{\equiv \hat{H}} \Psi(x, t) \tag{7}$$

a) Show that the wave function:

$$\Psi(x) = \frac{1}{\sqrt{x_0\sqrt{\pi}}} e^{-x^2/2x_0^2} \text{ with } x_0 = \sqrt{\hbar/m\omega} \tag{8}$$

is a solution of the *stationary* Schrödinger Eq.

$$\hat{H}\Psi(x) = E\Psi(x) \text{ with } E = \hbar\omega/2 \tag{9}$$

E corresponds to the *energy eigenvalue.*

b) Compute $\Psi(x,t)$.

c) Compute $\langle\hat{H}\rangle$, $\langle\hat{P}\rangle$ and $\langle\hat{X}\rangle$ for a particle in this state. Can you understand this result classically?

d) Is it possible to find a wave function with smaller energy? Hint. Use the uncertainty relation $\Delta x \Delta p \geq \hbar/2$.