QM1 Exercises. Sheet 4.

Corrections May 29-30

1) General Questions: Phases, interference and density matrices. (5 points)

a) Consider a wave function

$$|\Psi(t)\rangle = e^{i\phi}|\Psi_1(t)\rangle \tag{1}$$

Is it possible to devise an experiment which will measure ϕ? Give an argument for the result.

b) Now consider the wave function.

$$|\Psi(t)\rangle = |\Psi_1(t)\rangle + e^{i\phi}|\Psi_2(t)\rangle. \tag{2}$$

Does the result of a measurement depend on ϕ? Again give an argument for the result.

c) Let the density matrix of a mixed state be given by:

$$\hat{\rho} = \frac{1}{2}|\Psi_1\rangle\langle\Psi_1| + \frac{1}{2}|\Psi_2\rangle\langle\Psi_2|. \tag{3}$$

We assume that $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are normalized and orthogonal. Is it possible to find a state in the Hilbert space, $|\Psi\rangle$, which is a linear combination (or coherent superposition) of $|\Psi_1\rangle$ and $|\Psi_2\rangle$ such that $\hat{\rho} = |\Psi\rangle\langle\Psi|$? Justify your answer.

d) For the density matrix given in Eq. (3) compute the expectation value of the operator $\hat{\boldsymbol{O}} = |\boldsymbol{x}\rangle\langle\boldsymbol{x}|$. Compute the expectation value $\hat{O}$ in the state $|\Psi\rangle = \frac{1}{\sqrt{2}}|\Psi_1\rangle + \frac{1}{\sqrt{2}}|\Psi_2\rangle$. What is the physical difference between both results?

2) (5 points)

Consider a beam of atoms. Each atom has a two-fold internal degree of freedom which corresponds to the quantum states $|1\rangle$ or $|2\rangle$. (For example a spin 1/2 particle). We assume: $\langle n|n'\rangle = \delta_{n,n'}$ $(n, n' = 1, 2)$ and $\sum_{n=1}^{2} |n\rangle\langle n| = 1$. In this basis, the so-called Pauli operators $\hat{\sigma_x}$, $\hat{\sigma_y}$, and $\hat{\sigma_z}$ have matrix representations given by:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

a) Suppose that we have an apparatus, detector, which measures the observable

$$\boldsymbol{m} \cdot \hat{\boldsymbol{\sigma}} \equiv m_x\hat{\sigma}_x + m_y\hat{\sigma}_y + m_z\hat{\sigma}_z \tag{5}$$

with $\boldsymbol{m} \in \mathbb{R}^3$ and $|\boldsymbol{m}| = 1$. When the beam of atoms passes through the detector, show that the possible outcomes of the measurement are $\lambda_{\boldsymbol{m}} = \pm 1$?

b) If all the atoms in the beam are initially in state $|1\rangle$ what is the probability of obtaining the result $+1$, and what is the probability of obtaining the result -1.

c) Suppose that the beam of atoms goes through three detectors as follows.

i) The first detector measures $\hat{\sigma_z}$ and lets only the atoms with $\lambda_{\boldsymbol{e}_z} = 1$ through.

ii) The second detector measures $\boldsymbol{m} \cdot \hat{\boldsymbol{\sigma}}$ and lets the only the atoms with $\lambda_{\boldsymbol{m}} = 1$ through.

iii) The third detector measures $\hat{\sigma_z}$ and lets only atoms with $\lambda_{\boldsymbol{e}_z} = -1$ through.

If the intensity of the beam after the first measurement is equal to unity, what it the intensity of the final beam? How should you choose $\boldsymbol{m}$ so as to maximize the intensity of the final beam.

3) Landau levels (5 points)

Consider an electron confined in the $x - y$ plane in a uniform transverse magnetic field. The Hamiltonian of this system (in cgs units) is given by:

$$\hat{H} = \frac{1}{2m}\left(\hat{\boldsymbol{P}} - \frac{e}{c}\boldsymbol{A}(\hat{\boldsymbol{X}})\right)^2, \quad \hat{\boldsymbol{P}} = \left(\hat{P}_x, \hat{P}_y, 0\right), \quad \boldsymbol{\nabla} \times \boldsymbol{A}(x) = (0, 0, B). \tag{6}$$

a) Show that the kinetic momenta $\hat{\Pi}_i = \hat{\boldsymbol{P}}_i - \frac{e}{c}\boldsymbol{A}_i(\hat{\boldsymbol{X}})$ satisfy the commutation rules:

$$\left[\hat{\Pi}_x, \hat{\Pi}_y\right] = -\frac{\hbar e B}{ic} \tag{7}$$

b) Consider the operators

$$\hat{a}^+ = \sqrt{\frac{c}{2e\hbar B}}\left(\hat{\Pi}_x - i\hat{\Pi}_y\right) \quad \hat{a} = \sqrt{\frac{c}{2e\hbar B}}\left(\hat{\Pi}_x + i\hat{\Pi}_y\right). \tag{8}$$

We assume that $eB > 0$. Show that the commutation rules $[\hat{a}, \hat{a}^+] = 1$ hold.

c) Write the Hamiltonian in terms of the $\hat{a}^+$ and $\hat{a}$ operators and compare your result to that of the Harmonic oscillator. What is the energy spectrum?

The discrete energy levels are coined Landau levels and are at the origin of the so-called integer quantum Hall effect for which K. Von Klitzing obtained the Nobel prize in 1985.