QM1 Exercises. Sheet 8.

Corrections July 3-4

1) Rotation operator and angular momentum. (10 points)

a) Let $R(\boldsymbol{e}, \theta)$ denote rotation through an angle θ around the axis $\boldsymbol{e}$ ($|\boldsymbol{e}| = 1$). The direction of rotation follows the right hand rule. Show that:

$$R(\boldsymbol{e}, \theta)\boldsymbol{x} = \boldsymbol{e}(\boldsymbol{e} \cdot \boldsymbol{x}) - \cos(\theta)\left[\boldsymbol{e} \times (\boldsymbol{e} \times \boldsymbol{x})\right] + \sin(\theta)(\boldsymbol{e} \times \boldsymbol{x}) \tag{1}$$

A graphical proof suffices. $R(\boldsymbol{e}, \theta)$ corresponds of course to an $SO(3)$ matrix

b) Now show that:

$$\frac{d}{d\theta}\boldsymbol{x}(\theta) = \boldsymbol{e} \times \boldsymbol{x}(\theta) \quad \text{with} \quad \boldsymbol{x}(\theta) = R(\boldsymbol{e}, \theta)\boldsymbol{x} \tag{2}$$

c) Use the commutation rules of the angular momentum, $\hat{\boldsymbol{L}}$, and momentum, $\hat{\boldsymbol{P}}$, operators to show that:

$$\frac{d}{d\theta}\hat{\boldsymbol{P}}(\theta) = \boldsymbol{e} \times \hat{\boldsymbol{P}}(\theta) \quad \text{with} \quad \hat{\boldsymbol{P}}(\theta) = e^{i\boldsymbol{e}\cdot\hat{\boldsymbol{L}}\theta/\hbar}\hat{\boldsymbol{P}}e^{-i\boldsymbol{e}\cdot\hat{\boldsymbol{L}}\theta/\hbar} \tag{3}$$

In conjunction with **b)** you have shown that:

$$e^{i\boldsymbol{e}\cdot\hat{\boldsymbol{L}}\theta/\hbar}\hat{\boldsymbol{P}}e^{-i\boldsymbol{e}\cdot\hat{\boldsymbol{L}}\theta/\hbar} = R(\boldsymbol{e}, \theta)\hat{\boldsymbol{P}} \tag{4}$$

d) Show that

$$\hat{T}_{\boldsymbol{e},\theta} = e^{-i\boldsymbol{e}\cdot\hat{\boldsymbol{L}}\theta/\hbar} \tag{5}$$

is a unitary operator and that $\hat{T}_{\boldsymbol{e},\theta}|\boldsymbol{p}\rangle = |R(\boldsymbol{e}, \theta)\boldsymbol{p}\rangle$.

e) Repeat the same calculation to show that:

$$e^{i\boldsymbol{e}\cdot\hat{L}\theta/\hbar}\hat{\boldsymbol{X}}e^{-i\boldsymbol{e}\cdot\hat{L}\theta/\hbar} = R(\boldsymbol{e}, \theta)\hat{\boldsymbol{X}}, \quad \hat{T}_{\boldsymbol{e},\theta}|\boldsymbol{x}\rangle = |R(\boldsymbol{e}, \theta)\boldsymbol{x}\rangle. \tag{6}$$

2) Conservation of angular momentum for central potentials. (5 points)

Consider the Hamilton operator:

$$\hat{H}(\hat{\boldsymbol{P}}, \hat{\boldsymbol{X}}) = \frac{\hat{\boldsymbol{P}}^2}{2m} + V(\hat{\boldsymbol{X}}) \quad \text{with} \quad V(\boldsymbol{x}) = V(|\boldsymbol{x}|) \tag{7}$$

a) Show that

$$\hat{T}^{\dagger}_{\boldsymbol{e},\theta}\hat{H}\left(\hat{\boldsymbol{P}}, \hat{\boldsymbol{X}}\right)\hat{T}_{\boldsymbol{e},\theta} = \hat{H}\left[R(\boldsymbol{e}, \theta)\hat{\boldsymbol{P}}, R(\boldsymbol{e}, \theta)\hat{\boldsymbol{X}}\right] = \hat{H}(\hat{\boldsymbol{P}}, \hat{\boldsymbol{X}}) \tag{8}$$

b) Let $\hat{\boldsymbol{L}}(t)$ be the Heisenberg representation of the angular momentum. Show that:

$$\frac{d}{dt}\hat{\boldsymbol{L}}(t) = 0. \tag{9}$$

Hence, angular momentum is conserved for rotationally invariant Hamiltonians.