Piketty´s *Second Law of Capitalism*

Formal derivation, time dependent formulation, case discussion, validity and range of practical applicability

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Abstract

It is shown that Piketty´s *Second Law of Capitalism* (the Capital to Income ratio will have a finite limit value equal to savings rate *s* divided by growth rate *g*) is a mathematical consequence of the definition of Savings as an Integral over Income. It is formally correct for exponential growth with constant rates of growth and saving. No other growth function leads to a finite limit *s/g*. The time function for the transition from initial to final state is derived by differential calculus. It is characterized by exponentially decaying factors for both the Initial Capital to Income ratio and for the deviation from the final limit value. Cases of positive, negative and zero growth are discussed. Zero growth does not present a real world divergence problem of the ratio, as discussed in the literature; it leads to an unlimited, steady increase in time. A realm of plausible parameters for scenarios is calculated, which suggest that low growing rates should not be argued using limit values. An Excel simulation of the time dependent Capital to Income ratio is attached.

Introduction

In his important book *Capital in the Twenty-First Century*[1] Thomas Piketty investigates the change of Capital and Income over long time periods. In chapter V (page 166 onward of the English edition) a *Second Law of Capitalism* is introduced as

$$\beta = \frac{Capital}{Income} \overset{t\to\infty}{\rightarrow} \frac{s}{g} = \frac{savings\ rate}{growth\ rate\ of\ \ income}$$

In the long run the Capital to Income ratio β should approach a constant value that is equal to the ratio of savings rate to income growth rate, both of which are assumed to be constant.

Piketty emphasises that this is a long range law and that it may take many years before it is valid. He does not give a formal derivation of the law or a formulation of its time dependence. It stays unclear if it is an empirical rule or can be derived formally from basic assumptions.

The topic has created a tremendous discussion. At 27.1.2016 searching in Google under "Piketty second law" yields 170.000 links. A considerable number of the scientific contributions doubt the economic model of saving used, and/or treat technical questions concerning the formulation of the second law. It is criticized that the capital to income ratio "explodes" as growth rate approaches zero (an argument that Piketty uses himself), and a problematic divergence is seen when it is zero[2]. No sound mathematical model of Piketty´s

[1] Thomas Piketty *Capital in the Twenty-First Century*, The Bellknap Press of Harvard University 2014, ISBN 978-0-674-43000-6

[2] For example out of many http://aida.wss.yale.edu/smith/piketty1old2.pdf
Is Piketty's "Second Law of Capitalism" Fundamental? Per Krusell, Institute for International Economic Studies, CEPR, and NBER, Anthony A. Smith, Jr. Yale University and NBER October 21, 2014
http://georgecooper.org/2014/04/29/does-pikettys-r-g-hold-in-a-low-growth-world/
The Magical Mathematics of Mr Piketty Georg Cooper

second law seems to be common which includes the time development from the initial state to the limit value. Wrong conclusions may be drawn without proper consideration of the time dependence concerned.

Leaving aside the discussion about the economic model of income and saving, I concentrate on the mathematical problem of deriving a time dependent model of the capital to income ratio for a plausible definition of savings. If there are good reasons for preferring a different definition, the results could be easily adopted.

It will be shown that the assumption of uniform exponential income growth with constant savings rate leads to Piketty´s second law.

It will be shown that no other growth function has a limit value s/g.

The transition in time to the limit value is modelled in an Excel sheet, which allows visualizing the dependence on initial values, growth and savings rate. The time dependence of the C/I-ratio and its numerical presentation demonstrates that very low growth rate leads to very high transition times, which are beyond the realm of constant growth that can be assumed in practice. A plausible range of growth and savings rate is calculated that can be applied to realistic scenarios.

The problem is treated by differential calculus and differential rates. In the appendix it is shown what changes if yearly rates are used.

1.) Definitions

t : variable *time*

$C(t)$: Capital

$C(0)$: initial Capital at $t = 0$

$I(t)$: Income

$I(0)$: Initial Income

$s(t)$: differential savings rate

$$s(t) = \frac{1}{I(t)}\frac{dC(t)}{dt} \rightarrow \frac{\mathrm{d}C(t)}{dt} = s(t)I(t) \rightarrow$$

$S(t)$: Savings

$$S(t) = \int_0^t s(t)I(t)dt$$

$$C(t) = C(0) + S(t) = C(0) + \int_0^t s(t)I(t)dt$$

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http://marginalrevolution.com/marginalrevolution/2014/05/is-pikettys-second-law-of-capitalism-fundamental.html
Is Piketty's "Second Law of Capitalism" fundamental? Tyler Cowan

2.) General Formulation

The last equation leads to the general formula for the time dependence of the ratio of Capital to Income

$$\frac{C(t)}{I(t)} = \frac{C(0) + \int_0^t \mathrm{s}(t) I(t) dt}{I(t)} = \frac{C(0)}{I(t)} + \frac{\int_0^t \mathrm{s}(t) I(t) dt}{I(t)}$$

In it both growth rate and savings rate may vary in time (it may be reasonable to assume the savings rate depending on income). The relation can be easily investigated numerically, but cannot be further resolved analytically.

3.) Derivation of a Growth functions consistent with Piketty´s second law

To arrive at analytical solutions we assume the savings rate and the growth pattern to be constant, as Piketty does. We do not yet assume as specific growth function (e.g. linear or exponential) but will prove which growth function leads to a finite limit value consistent with Piketty´s second law

Assumptions

$s(t) = s = const$

$g(t) = g = const$ a parameter characterizing growth

Pikettys Second Law

1.) $\lim\limits_{t \to \infty} \dfrac{C(0)}{I(t)} = 0$

2.) $\lim\limits_{t \to \infty} \dfrac{S(t)}{I(t)} = const = \dfrac{s}{g}$

Derivation

Which function $I(t)$ obeys $\lim\limits_{t \to \infty} \dfrac{S(t)}{I(t)} = \lim\limits_{t \to \infty} \dfrac{s\int_0^t I(t) dt}{I(t)} = \dfrac{s}{g}$ **?**

The mathematically skilled will recognize that an exponential growth function fulfils the requirement: its integral is equal to the function itself except of a constant in the exponential:

$$I(t) = e^{gt}; \quad \int e^{gt} dt = \frac{1}{g} e^{gt}$$

With the following proof it is excluded that any other growth function fulfils the condition, and the full time dependence of the transition from the initial state to the final one is developed.

$$\lim_{t \to \infty} \frac{S(t)}{I(t)} = \lim_{t \to \infty} \frac{s\int_0^t I(t) dt}{I(t)} = \frac{s}{g} \to \lim_{t \to \infty} \left[\int_0^t I(t) dt = \frac{1}{g} I(t) \right]$$

We assume that *I(t)* is differentiable (has no kinks) and differentiate both sides of the equation

$$\lim_{t\to\infty}\left[I(t)=\frac{1}{g}\frac{dI(t)}{dt}\right]$$

$$\lim_{t\to\infty}\left[\frac{dI(t)}{I(t)}=gdt\right]$$

This is the well known differential equation of the exponential function <u>and of it only</u>.

It is valid for all values of the variable *t*, and we solve the differential equation by integration:

$$\int\frac{dI(x)}{I(x)}=\int gdx\rightarrow\ln I(t)=gt+const$$

Calculating the definite integral between the boundaries $0\leq x\leq t$ determines the integration constant, with the result

$$\int_0^t\frac{dI(x)}{I(x)}=\int_0^t\ gdx\rightarrow\ln I(t)\text{ - }\ln I(0)=gt$$

$$I(t)=I(0)e^{gt}$$

$$g=\frac{1}{I(t)}\frac{dI(t)}{dt}$$

The only growth function formally consistent with Piketty´s second law it that of constant exponential growth; *g* is its differential growth rate.

The time dependent second law becomes

$$\frac{C(t)}{I(t)}=\frac{C(0)+s\int_0^t(I(t)dt}{I(t)}=\frac{C(0}{I(0)}e^{-gt}+\frac{s}{g}\frac{(e^{gt}-1)}{e^{gt}}$$

$$\frac{C(t)}{I(t)}=\frac{C(0}{I(0)}e^{-gt}+\frac{s}{g}(1-e^{-gt})$$

Note that the development in time depends only on the growth rate *g*.

4.) Case discussion

a.) Positive growth g >0

$$g>0$$

$$\lim_{t\to\infty}\frac{C(0)}{I(0)}e^{-gt}=0\quad q.e.d.\quad\lim_{t\to\infty}\frac{C(t)}{I(t)}=\lim_{t\to\infty}\frac{s}{g}(1-e^{-gt})=\frac{s}{g}\quad q.e.d.$$

$$\frac{s}{g}<\frac{C(0)}{I(0)}\rightarrow\frac{C(t)}{I(t)}\quad\text{decreases to }\frac{s}{g}$$

$$\frac{s}{g}=\frac{C(0)}{I(0)}\rightarrow\frac{C(t)}{I(t)}=\frac{s}{g}=const$$

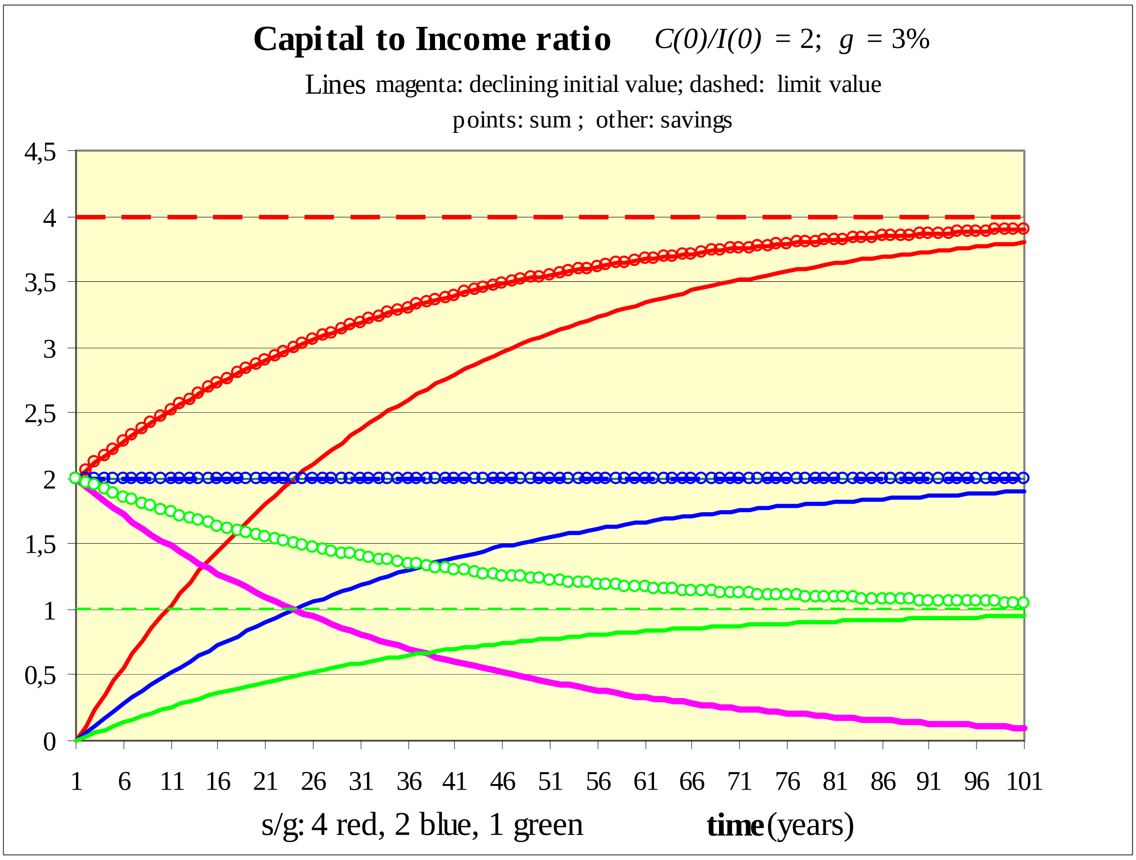
$$\frac{s}{g}>\frac{C(0)}{I(0)}\rightarrow\frac{C(t)}{I(t)}\quad\text{increases to }\frac{s}{g}$$

The relation of s/g to the initial *C(0)/I*(0) ratio determines whether *C(t)/I(t)* increases or decreases to an new limit value, or stays constant.

The time dependent law is programmed in an attached Excel sheet, where all parameters can be changed by sliders. The result is visualized in graphs. As example the following picture demonstrates the 3 cases discussed above for a uniform initial ratio of 2. The relevant curves are shown in different colours. The magenta curve of the declining impact of the initial ratio is uniform for all 3 cases, as it depends only on the initial value (2) and the growth rate (3%), both of which are held uniform. For

$$s/g = C(0)/I(0)$$

the sum curve (with points) and the limit line (dashed) coincide.



b.) Negative growth g<0

$$g < 0; g = -|g|$$

$$I(t) = I(0)\,e^{-|g|t} \to 0$$

$$\frac{C(t)}{I(t)} = e^{|g|t}\left(\frac{C(0)}{I(0)} - \frac{s}{g}\right) + \frac{s}{g}$$

$$\frac{C(t)}{I(t)} \to +\infty \quad for \quad \frac{s}{g} < \frac{C(0)}{I(0)}$$

$$\frac{C(t)}{I(t)} \to -\infty \quad for \quad \frac{s}{g} > \frac{C(0)}{I(0)}$$

The ratio goes to $\pm$ infinity with $e^{|g|t}$, as the denominator *I(t)* goes to zero.

c.) Zero growth rate g = 0

This case has aroused great dispute, as the *C/I* ratio seems to diverge, when g is set zero in the second law.

$$\lim_{g\to 0}\frac{C(t)}{I(t)}=\frac{s}{g}\to\infty ??$$

The application of this argument to real world problems is irrelevant. Two limit processes are involved: one in time, and one in *g*. To have an argument of practical applicability, both processes must be combined by using the time dependent formulation. It is done by developing the exponential into a time series, cancelling *g* in nominator and denominator, and then setting $g = 0$

$$e^{-gt}=1-gt+\frac{(gt)^2}{2}-...\to \underset{g\to 0}{e^{-gt}}=1$$

$$\frac{s}{g}(1-e^{-gt})=\frac{s}{g}(gt-\frac{(gt)^2}{2}+...)=s(t-\frac{gt^2}{2}+...)\underset{g\to 0}{\to} st$$

$$\frac{C(t)}{I(t)}=\frac{C(0)}{I(0)}+st \quad \text{steady increase in time}$$

For zero growth, with constant income, the ratio increases linearly in time beyond the initial one, when a constant part of income is saved. This appears self evident. How does this agree with the general assumption, that the ratio approaches a limit value (in this case infinity)? It would take an infinite time, and hence has no real world implication.

5.) A counterexample: linear growth

It was shown that only an exponential growth model is consistent with the second law.

As a counterexample we investigate linear growth in its familiar formulation with a growth rate δ (that is <u>not</u> the differential growth rate):

$$I(t)=I(0)(1+\delta t)$$

$$\delta=\frac{1}{I(0)}\frac{dI(t)}{dt} \quad \text{constant absolute increase per time interval}$$

$$\frac{S(t)}{I(t)}=\frac{s\int_0^t I(t)dt}{I(t)}=\frac{s\int_0^t (1+\delta t)dt}{(1+\delta t)}=\frac{s(t+\frac{\delta t^2}{2})}{(1+\delta t)}=\frac{st(1+\frac{\delta t}{2})}{(1+\delta t)}=st\frac{\frac{1}{t}+\frac{\delta}{2}}{\frac{1}{t}+\delta}$$

$$\frac{C(t)}{I(t)}=\frac{C(0)}{I(0)}(1+\delta t)^{-1}+st\frac{\frac{1}{t}+\frac{\delta}{2}}{\frac{1}{t}+\delta}$$

$$\delta=0\to\frac{C(t)}{I(t)}=\frac{C(0)}{I(0)}+st \quad \text{steady increase in time; no limit}$$

$$g=\frac{1}{I(t)}\frac{dI(t)}{dt}=\frac{\delta}{1+\delta t}$$

There is no limit value; the ratio stays increasing in time. The differential growing rate decreases with time.

Naturally the result is the same as for exponential growth with zero growth.

6.) Applicability of Piketty´s second law and useful range of parameters

The second law is consistent only with an exponential growth model. This is no very serious constriction, as exponential growth is a most natural assumption for limited time periods.

The restriction to constant growth rate weighs more. To stay within realistic time limits while arguing with the limit value, growth must not be too low.

The restriction to constant savings rate is less critical, as it enters into the value of the limit only and does not influence the transition in time.

A set of parameters is calculated, for which argumentation with limit values seems plausible

We set the following restrictions:

- The horizon of assumed constant growth should not be no longer than *T* years
- In this time *L* % of the limit values should be reached

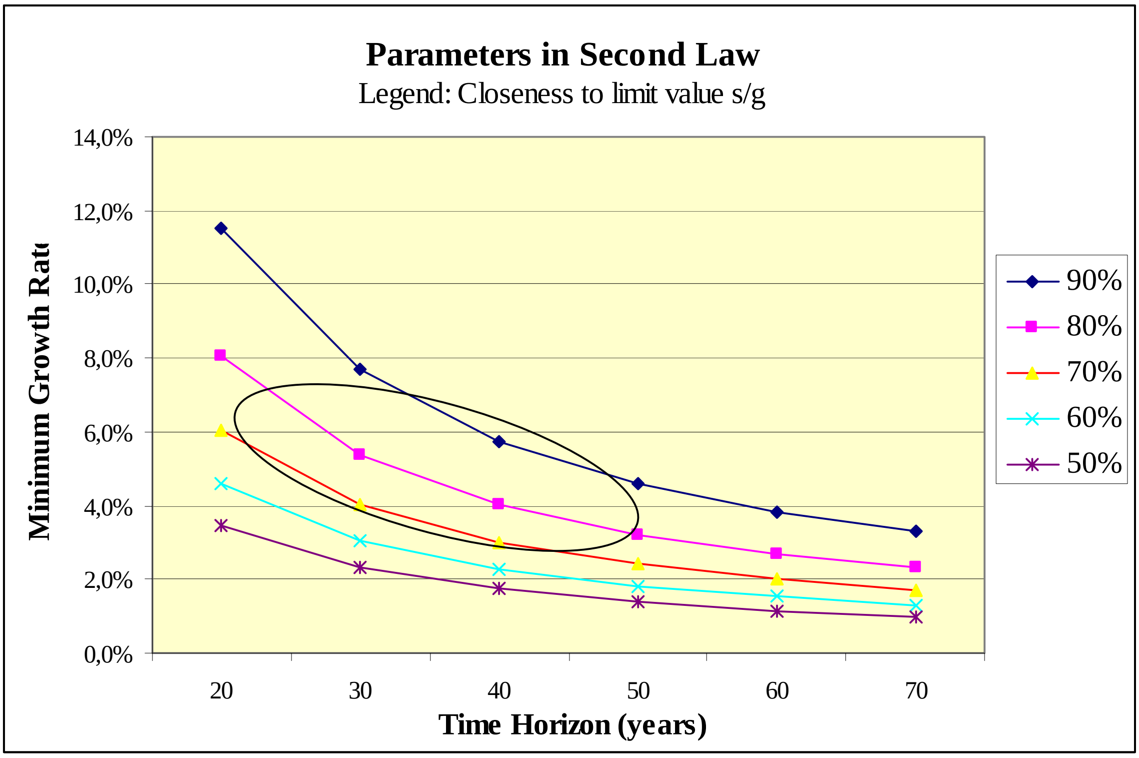
For these parameters we calculate necessary growth rates, which are the values in the following matrix.

$t \leq T$

$1 - e^{-gt} = L$

$g = -\frac{1}{T}\ln(1-L)$

T / L	20	30	40	50	60	70
90%	11,5%	7,7%	5,8%	4,6%	3,8%	3,3%
80%	8,0%	5,4%	4,0%	3,2%	2,7%	2,3%
70%	6,0%	4,0%	3,0%	2,4%	2,0%	1,7%
60%	4,6%	3,1%	2,3%	1,8%	1,5%	1,3%
50%	3,5%	2,3%	1,7%	1,4%	1,2%	1,0%



The set of curves in the picture shows the relation of growth rate and time for different closeness to the limit value (significance of its value). The ellipse encircles an area that to me seems plausible for argumentation. Really low growth rates are not within it as either the time horizon becomes to long, or the significance too low.

As a rule of thump, the reciprocal growth rate *1/g* is the time it takes for the exponential function to decay from a value of *1* to *1/e =34%*; it takes *2.3/g* to arrive at 10%.

All in all Piketty´s second law in its printed form seems more appropriate for qualitative than for quantitative argument – and that is really where he uses it most convincingly. Otherwise the complete formula for exponential growth with its time dependence should be used or the general formulation for arbitrary growth and savings pattern – not really a problem for building scenarios with today's PCs.

Large and fast changes in the distribution of capital in a society can not be explained by shifts of the total society, as its growth rate of income is too low. As Piketty stresses himself their analysis must consider different initial values, incomes and saving behaviour of different groups in the society. Such an analysis better starts with the general formulation of the Capital to Income rate, and investigates scenarios in numerical simulation. Such an analysis will be published separately.

Appendix

Which rates to use?

As appropriate for a differential analysis, differential rates of growth and of savings have been used throughout this text. They assume that income and savings flow continuously and describe the change in an infinitesimal time interval.

In economy one often one prefers to calculate in yearly rates, as in accounting practice.

Does Piketty´s law depend on the type of rate used?

The relation for the growth rates follows; numerical results are shown in the picture below

γ yearly growth rate

g differential growth rate

1.) formulation with differential growth rate $I(t) = I(0)e^{gt}$

2.) formulation with yearly growth rate $I(t) = I(0)(1+\gamma)^t$

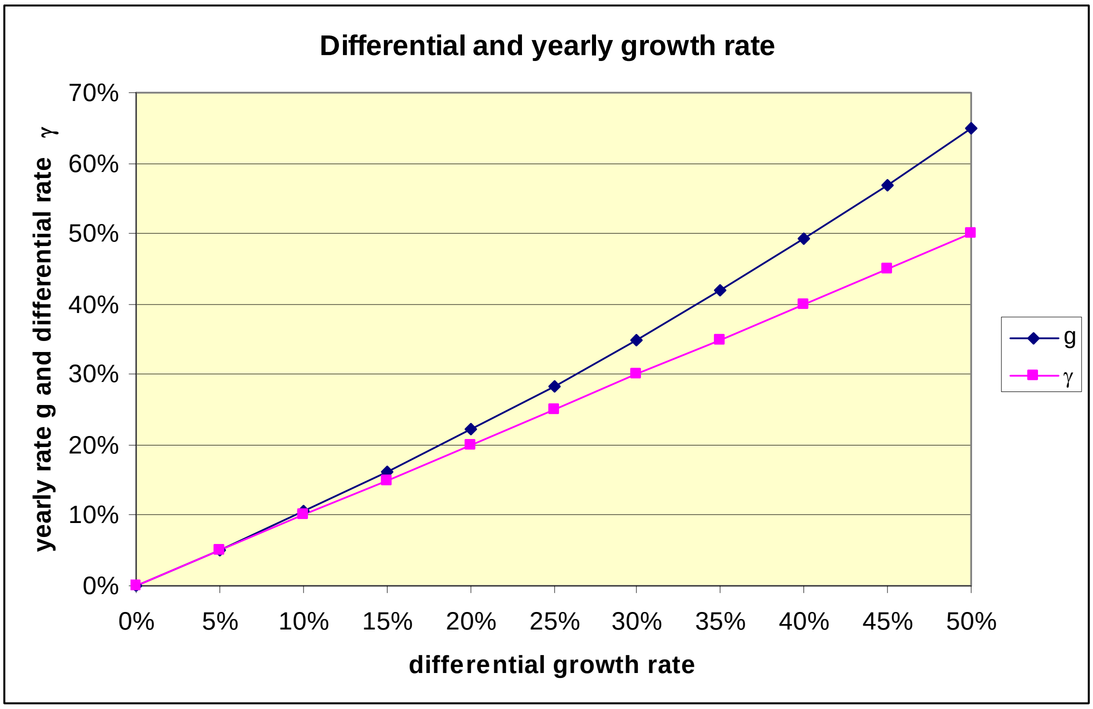
$\rightarrow (1+\gamma)^t = e^{gt}$

$g = \ln(1+\gamma)$

$\gamma = e^g - 1;$

$$1 \rightarrow \gamma \approx 1 + g + \frac{g^2}{2} + + + \quad -1$$

$$\gamma << 1 \rightarrow \gamma \approx g + \frac{g^2}{2}$$



The difference is substantial at high rates, but negligible at the low growth rates typical for income growth of nations within reasonable time periods.

The relation for the savings rates follows as

s :   differential savings rate

σ :   yearly savings rate

$\ln(1+\gamma) = \mathrm{g}$

1.) formulation with differential growth rate $I(t) = I(0)e^{gt} \rightarrow$

$$\frac{S(t)}{I(0)} = s\int_0^t I(t)dt = s\int_0^t e^{gt}dt = \frac{s}{g}(e^{gt} - 1)$$

2.) formulation with yearly growth rate $I(t) = I(0)(1+\gamma)^t = I(0)\,\mathrm{e}^{\ln(1+\gamma)t} \rightarrow$

$$\frac{S(t)}{I(0)} = \sigma\int_0^t (1+\gamma)^t dt = \sigma\int_0^t \mathrm{e}^{\ln(1+\gamma)t}dt = \frac{\sigma}{\ln(1+\gamma)}(\mathrm{e}^{\ln(1+\gamma)t} - 1)$$

$$\frac{\sigma}{\ln(1+\gamma)}(\mathrm{e}^{\ln(1+\gamma)t} - 1) = \frac{s}{g}(e^{gt} - 1) \quad \text{and} \quad \ln(1+\gamma) = \mathrm{g} \quad \rightarrow \frac{s}{g}(e^{gt} - 1) = \frac{\sigma}{g}(e^{gt} - 1)$$

$\sigma = s$

The savings rate to use is the same for differential and yearly growth rate.

Using the yearly growth rate γ the time dependent second law is

$$\frac{C(t)}{I(t)} = \frac{C(0}{I(0)}e^{-\ln(1+\gamma)t} + \frac{s}{\ln(1+\gamma)}(1 - e^{-\ln(1+\gamma)t})$$

The difference between both formulations is negligible at small rates and not too long times.

30.1.2016 Dieter Röß